

PSEUDO MV-ALGEBRAS AND LEXICOGRAPHIC PRODUCT

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ABSTRACT. We study algebraic conditions when a pseudo MV-algebra is an interval in the lexicographic product of an Abelian unital ℓ -group and an ℓ -group that is not necessary Abelian. We introduce (H, u) -perfect pseudo MV-algebras and strong (H, u) -perfect pseudo MV-algebras, the latter ones will have a representation by a lexicographic product. Fixing a unital ℓ -group (H, u) , the category of strong (H, u) -perfect pseudo MV-algebras is categorically equivalent to the category of ℓ -groups.

1. INTRODUCTION

MV-algebras were introduced by Chang [Cha] as the algebraic counterpart of Łukasiewicz infinite-valued calculus and during the last 56 years MV-algebras entered deeply in many areas of mathematics and logics. More than 10 years ago, a non-commutative generalization of MV-algebras has been independently appeared. These new algebras are said to be pseudo MV-algebras in [GeIo] or a generalized MV-algebras in [Rac]. For them author [Dvu2] generalized a famous Mundici's representation theorem, see e.g. [CDM, Cor 7.1.8], showing that every pseudo MV-algebra is always an interval in a unital ℓ -group not necessarily Abelian. Such algebras have the operation \oplus as a truncated sum and they have two negations. We note that the equality of these two negations does not necessarily imply that a pseudo MV-algebra is an MV-algebra. According to Komori's theorem [Kom], [CDM, Thm 8.4.4], the variety lattice of MV-algebras is countably, whereas the one of pseudo MV-algebras is uncountable, cf. [Jak, DvHo]. Therefore, the structure of pseudo MV-algebras is much richer than the one of MV-algebras. In [DvHo] it was shown that the class of pseudo MV-algebras where each maximal ideal is normal is a variety. This variety is also very rich and within this variety many important properties of MV-algebras remain.

In [DDT], perfect pseudo MV-algebras were studied. They are characterized as those that every element of a perfect pseudo MV-algebra is either an infinitesimal or a co-infinitesimal. In [DDT] we have shown that the category of perfect

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pseudo MV-algebras is equivalent to the variety of ℓ -groups, and every such an algebra M is in the form of an interval in the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$, i.e. $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$. This generalized the result from [DiLe1] for perfect MV-algebras. A more general structure, n -perfect pseudo MV-algebras were studied in [Dvu3]. They can be characterized as those pseudo MV-algebras that have $(n+1)$ -comparable slices, and their representation is again in the form of an interval in the lexicographic product $\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G$, i.e. every strong n -perfect pseudo MV-algebra M is of the form $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, where G is any ℓ -group. In the paper [Dvu4], we have studied so-called $(\mathbb{H}, 1)$ -perfect pseudo MV-algebras, where $(\mathbb{H}, 1)$ is a unital ℓ -subgroup of the unital ℓ -group of reals $(\mathbb{R}, 1)$. They can be represented in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$ and such MV-algebras were described in [DiLe2].

Recently, lexicographic MV-algebras were studied in [DFL]. Such algebras are of the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is an Abelian unital ℓ -group and G is an Abelian ℓ -group. The main aim of the present paper is to generalize such lexicographic MV-algebras also for the case of pseudo MV-algebras. Therefore, we introduce so-called (H, u) -perfect and strong (H, u) -perfect pseudo MV-algebras, where (H, u) is an Abelian unital ℓ -group. We show that strong (H, u) -perfect pseudo MV-algebras are always of the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$, where G is an ℓ -group not necessarily Abelian. This category will be always categorically equivalent with the variety of ℓ -groups. Therefore, we generalize many interesting results that were known only in the realm of MV-algebras, see [DiLe2, CiTo, DFL].

The paper is organized as follows. Section 2 gathers necessary properties of pseudo MV-algebras. Section 3 presents a definition of (H, u) -perfect pseudo MV-algebras as those which can be decomposed into a system of comparable slices indexed by the elements of the interval $[0, u]_H = \{h \in H : 0 \leq h \leq u\}$, where (H, u) is an Abelian unital ℓ -group. Section 4 defines strong (H, u) -perfect pseudo MV-algebras and we show their representation by $\Gamma(H \overrightarrow{\times} G, (u, 0))$. More details on local pseudo MV-algebras with retractive ideals will be done in Section 5. A free product representation of local pseudo MV-algebras will be done in Section 6. In Section 7 we describe pseudo MV-algebras with a so-called lexicographic ideal. A categorical equivalence of the category of strong (H, u) -perfect pseudo MV-algebras will be established in Section 8. Finally, in Section 9 we describe weak (H, u) -perfect pseudo MV-algebras as those that they can be represented in the form $\Gamma(H \overrightarrow{\times} G, (u, g))$, where g is an arbitrary element (not necessarily $g = 0$) of an ℓ -group G .

2. PSEUDO MV-ALGEBRAS

According to [GeIo], a *pseudo MV-algebra* or a GMV-algebra by [Rac] is an algebra $(M; \oplus, ^-, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation \odot defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

- (A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- (A2) $x \oplus 0 = 0 \oplus x = x$;
- (A3) $x \oplus 1 = 1 \oplus x = 1$;
- (A4) $1^\sim = 0$; $1^- = 0$;
- (A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

- (A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;²
 (A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
 (A8) $(x^-)^\sim = x$.

Any pseudo MV-algebra is a distributive lattice where (A6) and (A7) define the join $x \vee y$ and the meet $x \wedge y$ of x, y , respectively.

We note that a *po-group* (= partially ordered group) is a group $(G; +, 0)$ (written additively) endowed with a partial order \leq such that if $a \leq b$, $a, b \in G$, then $x + a + y \leq x + b + y$ for all $x, y \in G$. We denote by $G^+ = \{g \in G : g \geq 0\}$ the *positive cone* of G . If, in addition, G is a lattice under \leq , we call it an ℓ -group (= lattice ordered group). An element $u \in G^+$ is said to be a *strong unit* (= order unit) if $G = \bigcup_n [-nu, nu]$, and the couple (G, u) with a fixed strong unit u is said to be a *unital po-group* or a *unital ℓ -group*, respectively. The *commutative center* of a group H is the set $C(H) = \{h \in H : h + h' = h' + h, \forall h' \in H\}$. Finally, two unital ℓ -groups (G, u) and (H, v) is *isomorphic* if there is an ℓ -group isomorphism $\phi : G \rightarrow H$ such that $\phi(u) = v$. In a similar way an isomorphism and a homomorphism of unital po-groups are defined. For more information on po-groups and ℓ -groups and for unexplained notions about them, see [Fuc, Gla].

By \mathbb{R} and \mathbb{Z} we denote the groups of reals and natural numbers, respectively.

Between pseudo MV-algebras and unital ℓ -groups there is a very close connection: If u is a strong unit of a (not necessarily Abelian) ℓ -group G ,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then $(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$ is a pseudo MV-algebra [GeIo].

A pseudo MV-algebra M is an *MV-algebra* if $x \oplus y = y \oplus x$ for all $x, y \in M$. We denote by $\mathcal{P}_s\mathcal{MV}$ and \mathcal{MV} the variety of pseudo MV-algebras and MV-algebras, respectively.

The basic representation theorem for pseudo MV-algebras is the following generalization [Dvu2] of the Mundici's famous result:

Theorem 2.1. *For any pseudo MV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$, there exists a unique (up to isomorphism) unital ℓ -group (G, u) such that $(M; \oplus, ^-, ^\sim, 0, 1)$ is isomorphic to $(\Gamma(G, u); \oplus, ^-, ^\sim, 0, u)$. The functor Γ defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital ℓ -groups.*

We note that the class of pseudo MV-algebras is a variety whereas the class of unital ℓ -groups is not a variety because it is not closed under infinite products.

Due to this result, if $M = \Gamma(G, u)$ for some unital ℓ -group (G, u) , then M is linearly ordered iff G is a linearly ordered ℓ -group, see [Dvu1, Thm 5.3].

Besides a total operation \oplus , we can define a partial operation $+$ on any pseudo MV-algebra M in such a way that $x + y$ is defined iff $x \odot y = 0$ and then we set

$$x + y := x \oplus y. \quad (2.1)$$

² \odot has a higher binding priority than \oplus .

In other words, $x + y$ is precisely the group addition $x + y$ if the group sum $x + y$ is defined in M .

Let A, B be two subsets of M . We define (i) $A \leq B$ if $a \leq b$ for all $a \in A$ and all $b \in B$, (ii) $A \oplus B = \{a \oplus b : a \in A, b \in B\}$, and (iii) $A + B = \{a + b : \text{if } a + b \text{ exists in } M \text{ for } a \in A, b \in B\}$. We say that $A + B$ is *defined* in M if $a + b$ exists in M for each $a \in A$ and each $b \in B$. (iv) $A^- = \{a^- : a \in A\}$ and $A^\sim = \{a^\sim : a \in A\}$.

Using Theorem 2.1, we have if $y \leq x$, then $x \odot y^- = x - y$ and $y^\sim \odot x = -y + x$, where the subtraction $-$ is in fact the group subtraction in the representing unital ℓ -group.

We recall that if H and G are two po-groups, then the *lexicographic product* $H \overrightarrow{\times} G$ is the group $H \times G$ which is endowed with the lexicographic order: $(h, g) \leq (h_1, g_1)$ iff $h < h_1$ or $h = h_1$ and $g \leq g_1$. The lexicographic product $H \overrightarrow{\times} G$ is an ℓ -group iff H is linearly ordered group and G is an arbitrary ℓ -group, [Fuc, (d) p. 26]. If u is a strong unit for H , then $(u, 0)$ is a strong unit for $H \overrightarrow{\times} G$, and $\Gamma(H \overrightarrow{\times} G, (u, 0))$ is a pseudo MV-algebra.

We say that a pseudo MV-algebra M is *symmetric* if $x^- = x^\sim$ for all $x \in M$. The pseudo MV-algebra $\Gamma(G, u)$ is symmetric iff $u \in C(G)$, hence the variety of symmetric pseudo MV-algebras is a proper subvariety of the variety \mathcal{MV} . For example, $\Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$ is symmetric and it is an MV-algebra iff G is Abelian.

An *ideal* of a pseudo MV-algebra M is any non-empty subset I of M such that (i) $a \leq b \in I$ implies $a \in I$, and (ii) if $a, b \in I$, then $a \oplus b \in I$. An ideal is said to be (i) *maximal* if $I \neq M$ and it is not a proper subset of another ideal $J \neq M$; we denote by $\mathcal{M}(M)$ the set of maximal ideals of M , (ii) *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, and (iii) *normal* if $x \oplus I = I \oplus x$ for any $x \in M$; let $\mathcal{N}(M)$ be the set of normal ideals of M . A pseudo MV-algebra M is *local* if there is a unique maximal ideal and, in addition, this ideal also normal.

There is a one-to-one correspondence between normal ideals and congruences for pseudo MV-algebras, [GeIo, Thm 3.8]. The quotient pseudo MV-algebra over a normal ideal I , M/I , is defined as the set of all elements of the form $x/I := \{y \in M : x \odot y^- \oplus y \odot x^- \in I\}$, or equivalently, $x/I := \{y \in M : x^\sim \odot y \oplus y^\sim \odot x \in I\}$.

Let $x \in M$ and an integer $n \geq 0$ be given. We define

$$\begin{aligned} 0.x &:= 0, & 1 \odot x &:= x, & (n+1).x &:= (n.x) \oplus x, \\ x^0 &:= 1, & x^1 &:= x, & x^{n+1} &:= x^n \odot x, \\ 0x &:= 0, & 1x &:= x, & (n+1)x &:= (nx) + x, \end{aligned}$$

if nx and $(nx) + x$ are defined in M . An element x is said to be an *infinitesimal* if mx exists in M for every integer $m \geq 1$. We denote by $\text{Infin}(M)$ the set of all infinitesimals of M .

We define (i) the *radical* of a pseudo MV-algebra M , $\text{Rad}(M)$, as the set

$$\text{Rad}(M) = \bigcap \{I : I \in \mathcal{M}(M)\},$$

and (ii) the *normal radical* of M , $\text{Rad}_n(M)$, via

$$\text{Rad}_n(M) = \bigcap \{I : I \in \mathcal{N}(M) \cap \mathcal{M}(M)\}$$

whenever $\mathcal{N}(M) \cap \mathcal{M}(M) \neq \emptyset$.

By [DDJ, Prop. 4.1, Thm 4.2], it is possible to show that

$$\text{Rad}(M) \subseteq \text{Infin}(M) \subseteq \text{Rad}_n(M).$$

The notion of a state is an analogue of a probability measure for pseudo MV-algebras. We say that a mapping s from a pseudo MV-algebra M into the real interval is a *state* if (i) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in M , and (ii) $s(1) = 1$. We define the *kernel* of s as the set $\text{Ker}(s) = \{a \in M : s(a) = 0\}$. Then $\text{Ker}(s)$ is a normal ideal of M .

If M is an MV-algebra, then at least one state on M is defined. Unlike for MV-algebras, there are pseudo MV-algebras that are stateless, [Dvu1] (see also a note just before Theorem 8.5 below). We note that M has at least one state iff M has at least one maximal ideal that is also normal. However, every non-trivial linearly ordered pseudo MV-algebra admits a unique state, [Dvu1, Thm 5.5].

Let $\mathcal{S}(M)$ be the set of all states on M ; it is a convex set. A state s is said to be *extremal* if from $s = \lambda s_1 + (1 - \lambda)s_2$, where $s_1, s_2 \in \mathcal{S}(M)$ and $0 < \lambda < 1$, we conclude $s = s_1 = s_2$. Let $\partial_e \mathcal{S}(M)$ denote the set of extremal states. In addition, in view of [Dvu1], a state s is extremal iff $\text{Ker}(s)$ is a maximal ideal of M iff $s(a \wedge b) = \min\{s(a), s(b)\}$. Or equivalently, s is a *state morphism*, i.e., s is a homomorphism from M into the MV-algebra $\Gamma(\mathbb{R}, 1)$. In addition, a normal ideal I is maximal iff $I = \text{Ker}(s)$ for some extremal state s .

We say that a net of states $\{s_\alpha\}_\alpha$ *converges weakly* to a state s if $s(a) = \lim_\alpha s_\alpha(a)$, $a \in M$. Then $\mathcal{S}(M)$ and $\partial_e \mathcal{S}(M)$ are compact Hausdorff topological spaces, in particular cases both can be empty, and due to the Krein-Mil'man Theorem [Go, Thm 5.17], every state on M is a weak limit of a net of convex combinations of extremal states.

Pseudo MV-algebras can be studied also in the frames of pseudo effect algebra which are a non-commutative generalization of effect algebras introduced by [FoBe].

According to [DvVe1, DvVe2], a partial algebraic structure $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo effect algebra* if, for all $a, b, c \in E$, the following hold:

- (PE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$;
- (PE2) there are exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;
- (PE3) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;
- (PE4) if $a + 1$ or $1 + a$ exists, then $a = 0$.

If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then \leq is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. It is possible to show that $a \leq b$ if and only if $b = a + c = d + a$ for some $c, d \in E$. We write $c = a / b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a / b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a / 1$ for any $a \in E$.

If (G, u) is a unital po-group, then $(\Gamma(G, u); +, 0, u)$, where the set $\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$ is endowed with the restriction of the group addition $+$ to $\Gamma(G, u)$ and with 0 and u as 0 and 1, is a pseudo effect algebra. Due to [DvVe1, DvVe2], if a pseudo effect algebra satisfies a special type of the Riesz Decomposition Property, RDP_1 , then every pseudo effect algebra is an interval in some unique (up to isomorphism of unital po-groups) (G, u) satisfying also RDP_1 such that $M \cong \Gamma(G, u)$.

We say that a mapping f from one pseudo effect algebra E onto a second one F is a *homomorphism* if (i) $a, b \in E$ such that $a + b$ is defined in E , then $f(a) + f(b)$ is defined in F and $f(a + b) = f(a) + f(b)$, and (ii) $f(1) = 1$.

We say that a pseudo effect algebra E satisfies RDP_2 property if $a_1 + a_2 = b_1 + b_2$ implies that there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that (i) $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$ and $b_2 = c_{12} + c_{22}$, and (ii) $c_{12} \wedge c_{21} = 0$.

In [DvVe2, Thm 8.3, 8.4], it was proved that if $(M; \oplus, ^-, \sim, 0, 1)$ is a pseudo MV-algebra, then $(M; +, 0, 1)$, where $+$ is defined by (2.1), is a pseudo effect algebra with RDP_2 . Conversely, if $(E; +, 0, 1)$ is a pseudo effect algebra with RDP_2 , then E is a lattice, and by [DvVe2, Thm 8.8], $(E; \oplus, ^-, \sim, 0, 1)$, where

$$a \oplus b := (b^- \setminus (a \wedge b^-))^\sim, \quad (2.2)$$

is a pseudo MV-algebra. In addition, a pseudo effect algebra E has RDP_2 iff E is a lattice and E satisfies RDP_1 , see [DvVe2, Thm 8.8].

Finally, we note that an *ideal* of a pseudo effect algebra E is a non-empty subset I such that (i) if $x, y \in I$ and $x + y$ is defined in E , then $x + y \in I$, and (ii) $x \leq y \in I$ implies $x \in I$. An ideal I is *normal* if $a + I := \{a + i : i \in I \text{ if } a + i \text{ exists in } E\} = I + a := \{j + a : j \in I\}$ for any $a \in E$. A maximal ideal is defined in a standard way. If M is a pseudo MV-algebra, then the ideal I of M is also an ideal when M is understood as a pseudo effect algebra; this follows from the fact $x \oplus y = (x \wedge y^-) + y$.

We note that a mapping from a pseudo effect algebra E into a pseudo effect algebra F is a *homomorphism* if (i) $a + b \in E$ implies $h(a) + h(b) \in F$ and $h(a + b) = h(a) + h(b)$, and (ii) $h(1) = 1$. A bijective mapping $h : E \rightarrow F$ is an *isomorphism* if both h and h^{-1} are homomorphisms of pseudo effect algebras.

3. (H, u) -PERFECT PSEUDO MV-ALGEBRAS

Generalizing ideas from [DiLe1, DDT, Dvu3, Dvu4], we introduce the basic notions of our paper.

If (H, u) is a unital ℓ -group, we set $[0, u]_H := \{h \in H : 0 \leq h \leq u\}$.

Definition 3.1. Let (H, u) be a linearly ordered group and let u belong to the commutative center $C(H)$ of H . We say that a pseudo MV-algebra M is (H, u) -*perfect*, if there is a system $(M_t : t \in [0, u]_H)$ of nonempty subsets of M such that it is an (H, u) -*decomposition* of M , i.e. $M_s \cap M_t = \emptyset$ for $s < t$, $s, t \in [0, u]_H$ and $\bigcup_{t \in [0, u]_H} M_t = M$, and

- (a) $M_s \leq M_t$ for all $s < t$, $s, t \in [0, u]_H$;
- (b) $M_t^- = M_{u-t} = M_t^\sim$ for each $t \in [0, u]_H$;
- (c) if $x \in M_v$ and $y \in M_t$, then $x \oplus y \in M_{v \oplus t}$, where $v \oplus t = \min\{v + t, u\}$.

We note that (a) can be written equivalently in a stronger way: if $s < t$ and $a \in M_s$ and $b \in M_t$, then $a < b$. Indeed, by (b) we have $a \leq b$. If $a = b$, then $a \in M_s \cap M_t = \emptyset$, which is absurd. Hence, $a < b$.

In addition, (i) if $(H, u) = (\mathbb{Z}, 1)$ and M is an MV-algebra, we are speaking on a *perfect* MV-algebra, [DiLe1], (ii) if $(\mathbb{H}, u) = (\frac{1}{n}\mathbb{Z}, 1)$, a $(\frac{1}{n}\mathbb{Z}, 1)$ -perfect pseudo MV-algebra is said to be *n-perfect*, see [Dvu3], (iii) if \mathbb{H} is a subgroup of the group of real numbers \mathbb{R} , such that $1 \in \mathbb{H}$, $(\mathbb{H}, 1)$ -perfect pseudo MV-algebras are in [Dvu4] called *\mathbb{H} -perfect* pseudo MV-algebras.

For example, let

$$M = \Gamma(H \overrightarrow{\times} G, (u, 0)), \quad (3.1)$$

where $u \in C(H)$. We set $M_0 = \{(0, g) : g \in G^+\}$, $M_u := \{(u, -g) : g \in G^+\}$ and for $t \in [0, u]_H \setminus \{0, u\}$, we define $M_t := \{(t, g) : g \in G\}$. Then $(M_t : t \in [0, u]_H)$ is an (H, u) -decomposition of M and M is an (H, u) -perfect pseudo MV-algebra.

As a matter of interest, if O is the zero group, then $\Gamma(O \overrightarrow{\times} G, (0, 0))$ is a one-element pseudo MV-algebra. The pseudo MV-algebra $\Gamma(\mathbb{Z} \overrightarrow{\times} O, (1, 0))$ is a two-element Boolean algebra. In general, $\Gamma(H \overrightarrow{\times} O, (u, 0)) \cong \Gamma(H, u)$ and it is semisimple (that is, its radical is a singleton) iff H is Archimedean. If $G \neq O \neq H$, $\Gamma(H \overrightarrow{\times} G, (u, 0))$ is not semisimple.

Theorem 3.2. *Let $M = (M_t : t \in [0, u]_H)$ be an (H, u) -perfect pseudo MV-algebra.*

- (i) *Let $a \in M_v$, $b \in M_t$. If $v + t < u$, then $a + b$ is defined in M and $a + b \in M_{v+t}$; if $a + b$ is defined in M , then $v + t \leq u$. If $a + b$ is defined in M and $v + t = u$, then $a + b \in M_u$.*
- (ii) *$M_v + M_t$ is defined in M and $M_v + M_t = M_{v+t}$ whenever $v + t < u$.*
- (iii) *If $a \in M_v$ and $b \in M_t$, and $v + t > u$, then $a + b$ is not defined in M .*
- (iv) *If $a \in M_v$ and $b \in M_t$, then $a \vee b \in M_{v \vee t}$ and $a \wedge b \in M_{v \wedge t}$.*
- (v) *M admits a state s such that $M_0 \subseteq \text{Ker}(s)$.*
- (vi) *M_0 is a normal ideal of M such that $M_0 + M_0 = M_0$ and $M_0 \subseteq \text{Infin}(M)$.*
- (vii) *The quotient pseudo MV-algebra $M/M_0 \cong \Gamma(H, u)$.*
- (viii) *Let $M = (M'_t : t \in [0, u]_H)$ be another (H, u) -decomposition of M satisfying (a)–(c) of Definition 3.1, then $M_t = M'_t$ for each $t \in [0, u]_H$.*
- (ix) *M_0 is a prime ideal of M .*

Proof. (i) Assume $a \in M_v$ and $b \in M_t$. If $v + t < u$, then $b^- \in M_{u-t}$, so that $a \leq b^-$, and $a + b$ is defined in M . Conversely, let $a + b$ be defined, then $a \leq b^- \in M_{u-t}$. If $v + t > u$, then $v > u - t$ and $a > b^- \geq a$ which is absurd, and this gives $v + t \leq u$. Now let $v + t = u$ and $a + b$ be defined in M . By (c) of Definition 3.1, we have $a + b \in M_u$.

(ii) By (i), we have $M_v + M_t \subseteq M_{v+t}$. Suppose $z \in M_{v+t}$. Then, for any $x \in M_v$, we have $x \leq z$. Hence, $y = z - x$ is defined in M and $y \in M_w$ for some $w \in [0, u]_H$. Since $z = y + x \in M_{v+t} \cap M_{v+w}$, we conclude $t = w$ and $M_{v+t} \subseteq M_v + M_t$.

(iii) It follows from (i).

(iv) Inasmuch as $x \wedge y = (x \oplus y^\sim) - y^\sim$, we have by (c) of Definition 3.1, $(x \oplus y^\sim) - y^\sim \in M_s$, where $s = ((v + u - t) \wedge u) - (u - t) = v \wedge t$. Using a de Morgan law and property (d), we have $x \vee y \in M_{v \vee t}$.

(v) Let s_0 be a unique state on $\Gamma(H, u)$ which exists in view of [Dvu1, Thm 5.5]. Define a mapping $s : M \rightarrow [0, 1]$ by $s(x) = s_0(t)$ if $x \in M_t$. It is clear that s is a well-defined mapping. Take $a, b \in M$ such that $a + b$ is defined in M . Then there are unique indices v and t such that $a \in M_v$ and $b \in M_t$. By (i), $v + t \leq u$ and $a + b \in M_{v+t}$. Therefore, $s(a + b) = s_0(v + t) = s_0(v) + s_0(t) = s(a) + s(b)$. It is evident that $s(1) = 1$ and $M_0 \subseteq \text{Ker}(s)$.

(vi) It is clear that M_0 is an ideal of M . To prove the normality of M_0 , take $x \in M_v$ and $y \in M_0$. Then $x \oplus y = ((x \oplus y) - x) + x \in M_v$ which implies by (i)–(ii) $(x \oplus y) - x \in M_0$ and $x \oplus M_0 \subseteq M_0 \oplus x$. In the same way we proceed for the second implication.

Since $M_0 + M_0 = M_0$, by (ii) we have $M_0 \subseteq \text{Infin}(M)$.

(vii) Since by (vi) M_0 is a normal ideal, M/M_0 is a pseudo MV-algebra, too. Using (iv), it is easy to verify that $x \sim_{M_0} y$ iff there is an $h \in [0, u]_H$ such that $x, y \in M_h$. We define a mapping $\phi : M/M_0 \rightarrow \Gamma(H, u)$ by $\phi(x) = h$ iff $x \in M_h$ for some $h \in [0, u]_H$. The mapping ϕ is an isomorphism from M/M_0 onto $\Gamma(H, u)$.

(viii) Let $M = (M'_t : t \in [0, u]_H)$ be another (H, u) -decomposition of M . We assert $M_0 = M'_0$. If not, there are $x \in M_0 \setminus M'_0$ and $y \in M'_0 \setminus M_0$. By (a), we have $x < y$ as well as $y < x$ which is absurd. Hence, $M_0 = M'_0$. By (vi), M_0 is normal and by (vii), $M_0 \cong \Gamma(H, u) \cong M/M'_0$. If $x \sim_{M_0} y$, then $x, y \in M_h$ for some $h \in [0, u]_H$, as well as $x \sim_{M'_0} y$ implies $x, y \in M_{t'}$ and $t = t'$ which implies $M_t = M'_{t'}$ for any $t \in [0, 1]_H$.

(ix) By (vii), $M/M_0 \cong \Gamma(H, u)$, so that M/M_0 is a linear pseudo MV-algebra. Applying [Dvu1, Thm 6.1], we conclude that the normal ideal M_0 is prime. \square

Example 3.3. We define MV-algebras: $M_1 = \Gamma(\mathbb{Z} \overrightarrow{\times} (\mathbb{Z} \overrightarrow{\times} \mathbb{Z}), (1, (0, 0)))$, $M_2 = \Gamma((\mathbb{Z} \overrightarrow{\times} \mathbb{Z}) \overrightarrow{\times} \mathbb{Z}, ((1, 0), 0))$, and $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0, 0))$ which are mutually isomorphic. The first one is $(\mathbb{Z}, 1)$ -perfect, the second one is $(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ -perfect and of course, the linear unital ℓ -groups $(\mathbb{Z}, 1)$ and $(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ are not isomorphic while the first one is Archimedean and the second one is not Archimedean. Both pseudo MV-algebras define the corresponding natural $(\mathbb{Z}, 1)$ -decomposition $(M_t^1)_t$ and $(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ -decompositions $(M_q^2)_q$ of M_1 and M_2 , respectively. Then $M_0^1 = \{(0, (0, m)) : m \geq 0\} \cup \{(0, (n, m)) : n > 0, m \in \mathbb{Z}\} = \text{Ker}(s_1) = \text{Infin}(M_1)$, where s_1 is a unique state on M_1 ; it is two-valued. But $M_0^2 = \{((0, 0), m) : m \geq 0\} \subset \text{Ker}(s_2) = \text{Infin}(M_2)$, where s_2 is a unique state on M_2 , it vanishes only on $\text{Infin}(M_2)$; it is two-valued.

In what follows, we show that the normal ideal M_0 of an (H, u) -decomposition $(M_t : t \in [0, u]_H)$ of an (H, u) -perfect pseudo MV-algebra is maximal iff (H, u) is isomorphic with $(\mathbb{H}, 1)$, where \mathbb{H} is a subgroup of the group of reals \mathbb{R} with natural order, and $1 \in \mathbb{H}$.

Theorem 3.4. Let $(M_t : t \in [0, u]_H)$ be an (H, u) -decomposition of an (H, u) -perfect pseudo MV-algebra M . The following statements are equivalent:

- (i) M_0 is maximal.
- (ii) (H, u) is isomorphic to some $(\mathbb{H}, 1)$, where \mathbb{H} is a subgroup of the group \mathbb{R} and $1 \in \mathbb{H}$.
- (iii) M possesses a unique state s and $M_0 = \text{Ker}(s)$.

Proof. (i) \Rightarrow (ii). By [Dvu1, Prop 3.4-3.5], M_0 is maximal iff M/M_0 is simple, i.e. it does not contain any non-trivial proper ideal. By (vii) of Theorem 3.2, $(M/M_0, u/M_0) \cong \Gamma(H, u)$ which means by Theorem 2.1 that (H, u) is a linear, Archimedean and Abelian unital ℓ -group, and by Hölder's theorem, [Bir, Thm XIII.12] or [Fuc, Thm IV.1.1], it is isomorphic to some $(\mathbb{H}, 1)$, where \mathbb{H} is a subgroup of \mathbb{R} and $1 \in \mathbb{H}$.

(ii) \Rightarrow (iii). If $(H, u) \cong (\mathbb{H}, 1)$, where \mathbb{H} is a subgroup of \mathbb{R} and $1 \in \mathbb{H}$, then M is isomorphic to an $(\mathbb{H}, 1)$ -perfect pseudo MV-algebra. This kind of pseudo MV-algebras was studied in [Dvu4], and by [Dvu4, Thm 3.2(iv)], M possesses a unique state s . This state has the property $s(M) = \mathbb{H}$ and $\text{Ker}(s) = M_0$.

(iii) \Rightarrow (i). If s is a unique state of M and $M_0 = \text{Ker}(s)$, by [Dvu1], M_0 is a normal and maximal ideal of M . \square

Remark 3.5. We note that in Corollary 7.7 it will be shown that if an (H, u) -perfect pseudo MV-algebra M is of a stronger form, namely a strong (H, u) -perfect pseudo MV-algebra introduced in the next section, then M has a unique state. In general, the uniqueness of a state for any (H, u) -perfect pseudo MV-algebra is unknown.

We note that there is uncountably many non-isomorphic unital ℓ -subgroups $(\mathbb{H}, 1)$ of the unital group $(\mathbb{R}, 1)$. By [Go, Lem 4.21], every \mathbb{H} is either *cyclic*, i.e. $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ for some $n \geq 1$ or \mathbb{H} is dense in \mathbb{R} .

Therefore, if $\mathbb{H} = \mathbb{H}(\alpha)$ is a subgroup of \mathbb{R} generated by $\alpha \in [0, 1]$ and 1, then $\mathbb{H} = \frac{1}{n}\mathbb{Z}$ for some integer $n \geq 1$ if α is a rational number. Otherwise, $\mathbb{H}(\alpha)$ is countable and dense in \mathbb{R} , and $M(\alpha) := \Gamma(\mathbb{H}(\alpha), 1) = \{m + n\alpha : m, n \in \mathbb{Z}, 0 \leq m + n\alpha \leq 1\}$, see [CDM, p. 149]. Therefore, we have uncountably many non-isomorphic $(\mathbb{H}, 1)$ -perfect pseudo MV-algebras.

4. REPRESENTATION OF STRONG (H, u) -PERFECT PSEUDO MV-ALGEBRAS

In accordance with [Dvu4], we introduce the following notions and generalize results from [Dvu4] for strong (H, u) -perfect pseudo MV-algebras. Our aim is to find an algebraic characterization of pseudo MV-algebras that can be represented in the form of the lexicographic product

$$\Gamma(H \overrightarrow{\times} G, (u, 0)),$$

where (H, u) is a linearly ordered Abelian unital ℓ -group and G is an ℓ -group not necessarily Abelian. In [Dvu4], we have studied a particular case when $(H, u) = (\mathbb{H}, 1)$, where \mathbb{H} is a subgroup of reals.

We say that a pseudo MV-algebra M enjoys *unique extraction of roots of 1* if $a, b \in M$ and na, nb exist in M , and $na = 1 = nb$, then $a = b$. Every linearly ordered pseudo MV-algebra enjoys due to Theorem 2.1 and [Gla, Lem 2.1.4] unique extraction of roots. In addition, every pseudo MV-algebra $\Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a linearly ordered ℓ -group, enjoys unique extraction of roots of 1 for any $n \geq 1$ and for any ℓ -group G . Indeed, let $k(s, g) = (u, 0) = k(t, h)$ for some $s, t \in [0, u]_H$, $g, h \in G$, $k \geq 1$. Then $ks = u = kt$ which yields $s = t > 0$, and $kg = 0 = kh$ implies $g = 0 = h$.

Let $n \geq 1$ be an integer. An element a of a pseudo MV-algebra M is said to be *cyclic of order n* or simply *cyclic* if na exists in M and $na = 1$. If a is a cyclic element of order n , then $a^- = a^\sim$, indeed, $a^- = (n-1)a = a^\sim$. It is clear that 1 is a cyclic element of order 1.

Let $M = \Gamma(G, u)$ for some unital ℓ -group (G, u) . An element $c \in M$ such that (a) $nc = u$ for some integer $n \geq 1$, and (b) $c \in C(G)$, where $C(G)$ is a commutative center of G , is said to be a *strong cyclic element of order n* .

We note that if \mathbb{H} is a subgroup of reals and $t = 1/n$, then $c_{\frac{1}{n}}$ is a strong cyclic element of order n .

For example, the pseudo MV-algebra $M := \Gamma(\mathbb{Q} \overrightarrow{\times} G, (1, 0))$, where \mathbb{Q} is the group of rational numbers, for every integer $n \geq 1$, M has a unique cyclic element of order n , namely $a_n = (\frac{1}{n}, 0)$. The pseudo MV-algebra $\Gamma(\frac{1}{n}\mathbb{Z}, (1, 0))$ for a prime number $n \geq 1$, has the only cyclic element of order n , namely $(\frac{1}{n}, 0)$. If $M = \Gamma(G, u)$ and G is a representable ℓ -group, G enjoys unique extraction of roots of 1, therefore, M has at most one cyclic element of order n . In general, a pseudo MV-algebra M can have two different cyclic elements of the same order. But if M has a strong

cyclic element of order n , then it has a unique strong cyclic element of order n and a unique cyclic element of order n , [DvKo, Lem 5.2].

We say that an (H, u) -decomposition $(M_t : t \in [0, u]_H)$ of M has the *cyclic property* if there is a system of elements $(c_t \in M : t \in [0, u]_H)$ such that (i) $c_t \in M_t$ for any $t \in [0, u]_H$, (ii) if $v + t \leq 1$, $v, t \in [0, u]_H$, then $c_v + c_t = c_{v+t}$, and (iii) $c_1 = 1$. Properties: (a) $c_0 = 0$; indeed, by (ii) we have $c_0 + c_0 = c_0$, so that $c_0 = 0$. (b) If $t = 1/n$, then $c_{\frac{1}{n}}$ is a cyclic element of order n .

Let $M = \Gamma(G, u)$, where (G, u) is a unital ℓ -group. An (H, u) -decomposition $(M_t : t \in [0, u]_H)$ of M has the *strong cyclic property* if there is a system of elements $(c_t \in M : t \in [0, u]_H)$, called an (H, u) -*strong cyclic family*, such that

- (i) $c_t \in M_t \cap C(G)$ for each $t \in [0, u]_H$;
- (ii) if $v + t \leq 1$, $v, t \in [0, u]_H$, then $c_v + c_t = c_{v+t}$;
- (iii) $c_1 = 1$.

For example, let $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is an Abelian linearly ordered unital ℓ -group and G is an ℓ -group (not necessarily Abelian), and $M_t = \{(t, g) : (t, g) \in M\}$ for $t \in [0, u]_H$. If we set $c_t = (t, 0)$, $t \in [0, u]_H$, then the system $(c_t : t \in [0, u]_H)$ satisfies (i)–(iii) of the strong cyclic property, and $(M_t : t \in [0, u]_H)$ is an (H, u) -decomposition of M with the strong cyclic property.

Finally, we say that a pseudo MV-algebra M is *strong (H, u) -perfect* if there is an (H, u) -decomposition $(M_t : t \in [0, u]_H)$ of M with the strong cyclic property.

A prototypical example of a strong (H, u) -perfect pseudo MV-algebra is the following.

Proposition 4.1. *Let G be an ℓ -group and (H, u) an Abelian unital ℓ -group. Then the pseudo MV-algebra*

$$\mathcal{M}_{H,u}(G) := \Gamma(H \overrightarrow{\times} G, (u, 0)) \quad (4.1)$$

is a strong (H, u) -perfect pseudo MV-algebra with a strong cyclic family $((h, 0) : h \in [0, u]_H)$.

Now we present a representation theorem for strong (H, u) -perfect pseudo MV-algebras by (4.1). The following theorem uses the basic ideas of the particular situation $(H, u) = (\mathbb{H}, 1)$ which was proved in [Dvu4, Thm 4.3].

Theorem 4.2. *Let M be a strong (H, u) -perfect pseudo MV-algebra, where (H, u) is an Abelian unital linearly ordered ℓ -group. Then there is a unique (up to isomorphism) ℓ -group G such that $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$.*

Proof. Since M is a pseudo MV-algebra, due to [Dvu2, Thm 3.9], there is a unique unital (up to isomorphism of unital ℓ -groups) ℓ -group (K, v) such that $M \cong \Gamma(K, v)$. Without loss of generality we can assume that $M = \Gamma(K, v)$. Assume $(M_t : t \in [0, u]_H)$ is an (H, u) -decomposition of M with the strong cyclic property and with an (H, u) -strong cyclic family $(c_t \in M : t \in [0, u]_H)$.

By (vi) of Theorem 3.2, M_0 is an associative cancellative semigroup satisfying conditions of Birkhoff's Theorem [Bir, Thm XIV.2.1], [Fuc, Thm II.4], which guarantees that M_0 is a positive cone of a unique (up to isomorphism) directed po-group G . Since M_0 is a lattice, we have that G is an ℓ -group.

Take the (H, u) -strong perfect pseudo MV-algebra $\mathcal{M}_{H,u}(G)$ defined by (4.1), and define a mapping $\phi : M \rightarrow \mathcal{M}_{H,u}(G)$ by

$$\phi(x) := (t, x - c_t) \quad (4.2)$$

whenever $x \in M_t$ for some $t \in [0, u]_H$, where $x - c_t$ denotes the difference taken in the group K .

Claim 1: ϕ is a well-defined mapping.

Indeed, M_0 is in fact the positive cone of an ℓ -group G which is a subgroup of K . Let $x \in M_t$. For the element $x - c_t \in K$, we define $(x - c_t)^+ := (x - c_t) \vee 0 = (x \vee c_t) - c_t \in M_0$ (when we use (iii) of Theorem 3.2) and similarly $(x - c_t)^- := -((x - c_t) \wedge 0) = c_t - (x \wedge c_t) \in M_0$. This implies that $x - c_t = (x - c_t)^+ - (x - c_t)^- \in G$.

Claim 2: The mapping ϕ is an injective and surjective homomorphism of pseudo effect algebras.

We have $\phi(0) = (0, 0)$ and $\phi(1) = (1, 0)$. Let $x \in M_t$. Then $x^- \in M_{1-t}$, and $\phi(x^-) = (1 - t, x - c_{1-t}) = (1, 0) - (t, x - c_t) = \phi(x)^-$. In an analogous way, $\phi(x^\sim) = \phi(x)^\sim$.

Now let $x, y \in M$ and let $x + y$ be defined in M . Then $x \in M_{t_1}$ and $y \in M_{t_2}$. Since $x \leq y^-$, we have $t_1 \leq 1 - t_2$ so that $\phi(x) \leq \phi(y^-) = \phi(y)^-$ which means $\phi(x) + \phi(y)$ is defined in $\mathcal{M}_{H,u}(G)$. Then $\phi(x + y) = (t_1 + t_2, x + y - c_{t_1+t_2}) = (t_1 + t_2, x + y - (c_{t_1} + c_{t_2})) = (t_1, x - c_{t_1}) + (t_2, y - c_{t_2}) = \phi(x) + \phi(y)$.

Assume $\phi(x) \leq \phi(y)$ for some $x \in M_t$ and $y \in M_v$. Then $(t, x - c_t) \leq (v, y - c_v)$. If $t = v$, then $x - c_t \leq y - c_t$ so that $x \leq y$. If $i < j$, then $x \in M_t$ and $y \in M_v$ so that $x < y$. Therefore, ϕ is injective.

To prove that ϕ is surjective, assume two cases: (i) Take $g \in G^+ = M_0$. Then $\phi(g) = (0, g)$. In addition $g^- \in M_1$ so that $\phi(g^-) = \phi(g)^- = (0, g)^- = (1, 0) - (0, g) = (1, -g)$. (ii) Let $g \in G$ and t with $0 < t < 1$ be given. Then $g = g_1 - g_2$, where $g_1, g_2 \in G^+ = M_0$. Since $c_t \in M_t$, $g_1 + c_t$ exists in M and it belongs to M_t , and $g_2 \leq g_1 + c_t$ which yields $(g_1 + c_t) - g_2 = (g_1 + c_t) \setminus g_2 \in M_t$. Hence, $g + c_t = (g_1 + c_t) \setminus g_2 \in M_t$ which entails $\phi(g + c_t) = (t, g)$.

Claim 3: If $x \leq y$, then $\phi(y \setminus x) = \phi(y) \setminus \phi(x)$ and $\phi(x / y) = \phi(x) / \phi(y)$.

It follows from the fact that ϕ is a homomorphism of pseudo effect algebras.

Claim 4: $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$.

We have, $\phi(x), \phi(y) \geq \phi(x \wedge y)$. If $\phi(x), \phi(y) \geq \phi(w)$ for some $w \in M$, we have $x, y \geq w$ and $x \wedge y \geq w$. In the same way we deal with \vee .

Claim 5: ϕ is a homomorphism of pseudo MV-algebras.

It is necessary to show that $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$. This follows straightforward from the previous claims and equality (2.2).

Consequently, M is isomorphic to $\mathcal{M}_{H,u}(G)$ as pseudo MV-algebras.

If $M \cong \Gamma(H \overrightarrow{\times} G', (u, 0))$ for some G' , then $(H \overrightarrow{\times} G, (u, 0))$ and $(H \overrightarrow{\times} G', (u, 0))$ are isomorphic unital ℓ -groups in view of the categorical equivalence, see [Dvu2, Thm 6.4] or Theorem 2.1; let $\psi : \Gamma(H \overrightarrow{\times} G, (u, 0)) \rightarrow \Gamma(H \overrightarrow{\times} G', (u, 0))$ be an isomorphism of the lexicographic products. Hence, by Theorem 3.2(viii), we see that $\psi(\{(0, g) : g \in G^+\}) = \{(0, g') : g' \in G'^+\}$ which proves that G and G' are isomorphic ℓ -groups. \square

We say that a pseudo MV-algebra is *lexicographic* if there are an Abelian linearly ordered unital ℓ -group (H, u) and an ℓ -group G (not necessarily Abelian) such that $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$. In other words, by Theorem 4.2, M is lexicographic iff M is

strong (H, u) -perfect for some Abelian linear unital ℓ -group (H, u) . We note that in [DFL], a lexicographic MV-algebra denotes an MV-algebra having a lexicographic ideal which will be defined below in Section 7. But by Theorem 7.5, we will conclude that both notions are equivalent for symmetric pseudo MV-algebras from \mathcal{M} .

It is worthy to note that according to Example 3.3, the pseudo MV-algebra M has two isomorphic lexicographic representations $\Gamma(\mathbb{Z} \overrightarrow{\times} (\mathbb{Z} \overrightarrow{\times} \mathbb{Z}), (1, (0, 0)))$ and $\Gamma((\mathbb{Z} \overrightarrow{\times} \mathbb{Z}) \overrightarrow{\times} \mathbb{Z}, ((1, 0), 0))$, but $(H_1, u_1) := (\mathbb{Z}, 1)$ and $(H_2, u_2) := (\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0))$ are not isomorphic, as well as $G_1 := \mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ and $G_2 := \mathbb{Z}$ are not isomorphic ℓ -groups.

5. LOCAL PSEUDO MV-ALGEBRAS WITH RETRACTIVE RADICAL

In [DiLe2, Cor 2.4], the authors characterized MV-algebras that can be expressed in the form $\Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$, where \mathbb{H} is a subgroup of \mathbb{R} and G is an Abelian ℓ -group. In what follows, we extend this characterization for local symmetric pseudo MV-algebras. This result gives another characterization of strong $(\mathbb{H}, 1)$ -perfect pseudo MV-algebras via lexicographic product.

We denote by \mathcal{M} the set of pseudo MV-algebras M such that either every maximal ideal of M is normal or M is trivial. By [DDT, (6.1)], \mathcal{M} is a variety.

Let M be a symmetric pseudo MV-algebra. For any $x \in M$, we define the *order*, in symbols $\text{ord}(x)$, as the least integer n such that $n \cdot x = 1$ if such n exists, otherwise, $\text{ord}(x) = \infty$. It is clear that the set of all elements with infinite order is an ideal. An element x is *finite* if $\text{ord}(x) < \infty$ and $\text{ord}(x^-) < \infty$.

Lemma 5.1. *Let M be a pseudo MV-algebra from \mathcal{M} and $x \in M$. There exists a proper normal ideal of M containing x if and only if $\text{ord}(x) = \infty$.*

Proof. Let x be any element of M and let $I(x)$ be the normal ideal of M generated by x . Then

$$I(x) = \{y \in M : y \leq m \cdot x \text{ for some } m \in \mathbb{N}\}. \quad (5.1)$$

□

Lemma 5.2. *Let M be a symmetric pseudo MV-algebra. If $\text{ord}(x \odot y) < \infty$, then $x \leq y^-$.*

Proof. By the hypothesis, $\text{ord}(x \odot y) = n$ for some integer $n \geq 1$. Hence $(y^- \oplus x^-)^n = 0$. By [GeLo, Prop 1.24(ii)], $(y^- \oplus x^-) \vee (x \oplus y) = 1$ which by [GeLo, Lem 1.32] yields $(y^- \oplus x^-)^n \vee (x \oplus y)^n = 1$, so that $(x \oplus y)^n = 1$ and $x \oplus y = 1$, consequently, $x \leq y^-$. □

Lemma 5.3. *Let $M \in \mathcal{M}$ be a symmetric pseudo MV-algebra. The following statements are equivalent:*

- (i) M is local.
- (ii) For every $x \in M$, $\text{ord}(x) < \infty$ or $\text{ord}(x^-) < \infty$.

Proof. Let M be local. There exists a unique maximal ideal I that is normal. Assume that for some $x \in M$, we have $\text{ord}(x) = \infty = \text{ord}(x^-)$. By Lemma 5.1, $x, x^- \in I$ which is absurd.

Conversely, let for every $x \in M$, $\text{ord}(x) < \infty$ or $\text{ord}(x^-) < \infty$. Let I be a maximal ideal of M and assume that $x \notin M$ for some $x \in M$ with $\text{ord}(x) = \infty$. Since I is by the hypothesis normal, by a characterization of normal and maximal ideals, [GeLo, Prop 3.5], there is an integer $n \geq 1$ such that $(x^-)^n \in I$. By Lemma 5.1, $\text{ord}((x^-)^n) = \infty$ and $\text{ord}(((x^-)^n)^-) < \infty$, i.e. $\text{ord}(n \cdot x) < \infty$, which implies

$\text{ord}(x) < \infty$ that is impossible. Hence, every element x with infinite order belongs to I , and so I is a unique maximal ideal of M , in addition I is normal. \square

Lemma 5.4. *Let $M \in \mathcal{M}$ be a local symmetric pseudo MV-algebra and let I be a unique maximal ideal of M . For all $x, y \in M$ such that $x/I \neq y/I$, we have $x < y$ or $y < x$.*

Proof. By hypothesis, we have that $x \odot y^- \notin I$ or $y \odot x^- \notin I$. By Lemma 5.3, in the first case we have $\text{ord}(x \odot y^-) < \infty$ which by Lemma 5.2 implies $x \leq y$ and consequently $x < y$. In the second case, we similarly conclude $y < x$. \square

We introduce the following notion. A normal ideal I of a pseudo MV-algebra M is said to be *retractive* if the canonical projection $\pi_I : M \rightarrow M/I$ is retractive, i.e. there is a homomorphism $\delta_I : M/I \rightarrow M$ such that $\pi_I \circ \delta_I = \text{id}_{M/I}$. If a normal ideal I is retractive, then δ_I is injective and M/I is isomorphic to a subalgebra of M .

For example, if $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$ and $I = \{(0, g) : g \in G^+\}$, then I is a normal ideal, see Theorem 3.2(vi), and due to $M/I \cong \Gamma(H, u) \cong \Gamma(H \overrightarrow{\times} \{0\}, (u, 0)) \subseteq \Gamma(H \overrightarrow{\times} G, (u, 0))$, I is retractive.

Lemma 5.5. *Let I be a normal ideal of a symmetric pseudo MV-algebra. Then the following are equivalent:*

- (i) $x/I = y/I$.
- (ii) $x = (h \oplus y) \odot k^-$, where $h, k \in I$.

Proof. (i) \Rightarrow (ii) Assume $x/I = y/I$. Then the elements $k = x^- \odot y$ and $h = x \odot y^-$ belong to I . It is easy to see that $x \oplus k = x \vee y = h \oplus y$. Since $k^- = y^- \oplus x \geq x$, we have $x = x \wedge k^- = (x \oplus k) \odot k^- = (h \oplus y) \odot k^-$.

(ii) \Rightarrow (i) Then we have $x/I = y/I$. \square

Let M be a pseudo MV-algebra, and let $\text{Sub}(M)$ be the set of all subalgebras of M . Then $\text{Sub}(M)$ is a lattice with respect to set theoretical inclusion with the smallest element $\{0, 1\}$ and greatest one M . It is easy to see that if M is symmetric and I is an ideal of M , then the subalgebra $\langle I \rangle$ of M generated by I is the set $\langle I \rangle = I \cup I^-$. We recall a subalgebra S of M is said to be a *complement* of a subalgebra A of M if $S \cap A = \{0, 1\}$ and $S \vee A = M$.

In the following, we characterize retractive ideals of pseudo MV-algebras in an analogous way as it was done for MV-algebras in [CiTo, Thm 1.2].

Theorem 5.6. *Let M be a symmetric pseudo MV-algebra and I a normal ideal of M . The following statements are equivalent:*

- (i) I is a retractive ideal.
- (ii) $\langle I \rangle$ has a complement.

Proof. Let I be a retractive ideal of M and let $\delta_I : M/I \rightarrow M$ be an injective homomorphism such that $\pi_I \circ \delta_I = \text{id}_{M/I}$. We claim that $\delta_I(M/I)$ is a complement of $\langle I \rangle$. Clearly $\delta_I(M/I) \cap \langle I \rangle = \{0, 1\}$. Let $x \in M$, then $x/I = \delta_I(M/I)(x/I)/I$ so that by Lemma 5.5, we have $x = (h \oplus \delta_I(x/I)) \odot k^-$ for some $h, k \in I$ that implies $x \in \delta_I(M/I) \vee \langle I \rangle$.

Conversely, assume that $\langle I \rangle$ has a complement $S \in \text{Sub}(M)$. From $S \cap \langle I \rangle = \{0, 1\}$ we conclude that the canonical projection π_I is injective on S . Indeed, if for $x, y \in S$, we have $x/I = y/I$, then also $x/I = (x \vee y)/I = y/I$ which yields

$(x \vee y) \odot x^- \in S \cap I = \{0\}$, $(x \vee y) \odot y^- \in S \cap I = \{0\}$. Therefore, $x = x \vee y = y$ and this implies that the restriction $\pi_{I|S}$ is injective.

From $S \vee \langle I \rangle = M$, we have that for each $x \in M$, there is a term in the language of pseudo MV-algebras, says $p(a_1, \dots, a_m, b_1, \dots, b_n)$, such that

$$x = p^M(x_1, \dots, x_m, y_1, \dots, y_n)$$

for some $x_1, \dots, x_m \in S$ and $y_1, \dots, y_n \in \langle I \rangle$. Then

$$x/I = p^{M/I}(x_1/I, \dots, x_m/I, y_1/I, \dots, y_n/I).$$

Since $y_i/I \in \{0, 1\}$ for each $i = 1, \dots, n$, there is an n -tuple (t_1, \dots, t_n) of elements from $\{0, 1\}$ such that $x/I = p^M(x_1, \dots, x_m, t_1, \dots, t_n)/I$. Since

$$(x_1, \dots, x_m, t_1, \dots, t_n) \in S^{m+n},$$

we have that $p^M(x_1, \dots, x_m, t_1, \dots, t_n) \in S$. Therefore, the restriction $\pi_{I|S}$ is an isomorphism from S onto M/I , and setting $\delta_I = (\pi_{I|S})^{-1}$, we see that I is retractive. \square

Theorem 5.7. *Let M be a symmetric pseudo MV-algebra from \mathcal{M} . The following statements are equivalent:*

- (i) M is local and $\text{Rad}_n(M)$ is retractive.
- (ii) M is strong $(\mathbb{H}, 1)$ -perfect for some subgroup \mathbb{H} of \mathbb{R} with $1 \in \mathbb{H}$.
- (iii) There exists a subgroup \mathbb{H} of \mathbb{R} with $1 \in \mathbb{H}$ and an ℓ -group G such that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (u, 0))$.

Proof. (i) \Rightarrow (ii) Let I be a unique maximal and normal ideal of M and let (K, v) be a (unique up to isomorphism) unital ℓ -group given by Theorem 2.1, such that $M \cong \Gamma(K, v)$; without loss of generality we can assume that $M = \Gamma(K, v)$. By [Dvu1], there is an extremal state (= state morphism) $s_0 : M \rightarrow [0, 1]$ such that $I = \text{Ker}(s_0)$. The range of s_0 , $s_0(M)$, is an MV-algebra which corresponds to a unique subgroup \mathbb{H} of \mathbb{R} such that $s_0(M) = \Gamma(\mathbb{H}, 1)$ is a subalgebra of $\Gamma(\mathbb{R}, 1)$.

Since $I = \text{Rad}_n(M)$, I is a retractive ideal, and M/I is isomorphic to $\Gamma(\mathbb{H}, 1)$, we have $\Gamma(\mathbb{H}, 1)$ can be injectively embedded into K and \mathbb{H} is isomorphic to a subgroup of K .

In addition, let $\langle I \rangle$ be a subalgebra of M generated by I . Then $\langle I \rangle = I \cup I^- = I \cup I^-, I^- = I^\sim$, and $\langle I \rangle$ is a perfect pseudo MV-algebra. By [DDT, Prop 5.2], there is a unique (up to isomorphism) ℓ -group G such that $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.

In what follows, we prove that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$.

Define $M_t = s^{-1}(\{t\})$, $t \in [0, 1]_{\mathbb{H}}$. We assert that $(M_t : t \in [0, 1]_{\mathbb{H}})$ is an $(\mathbb{H}, 1)$ -decomposition of M . It is clear that it is a decomposition: Every M_t is non-empty, and $M_t^- = M_{1-t} = M_t^\sim$ for each $t \in [0, 1]_{\mathbb{H}}$. In addition, if $x \in M_v$ and $y \in M_t$, then $x \oplus y \in M_{v \oplus t}$, $x \wedge y \in M_{v \wedge t}$ and $x \vee y \in M_{v \vee t}$. By Lemma 5.4, we have $M_s \leq M_t$ for all $s < t$, $s, t \in [0, 1]_{\mathbb{H}}$.

Since $I = \text{Rad}_n(M)$ is retractive, there is a unique subalgebra M' of M such that $s_0(M') = s_0(M)$. For any $t \in [0, 1]_{\mathbb{H}}$, there is a unique element $x_t \in M'$ such that $s_0(x_t) = t$. We assert that the system $(x_t : t \in [0, 1]_{\mathbb{H}})$ satisfies the following properties (i) $c_t \in M_t$ for each $t \in [0, 1]_{\mathbb{H}}$, (ii) $c_{v+t} = c_v + c_t$ whenever $v + t \leq 1$, (iii) $c_1 = 1$. (iv) $c_t \in C(K)$. Indeed, since s_0 is a homomorphism of pseudo MV-algebras, by the categorical equivalence Theorem 2.1, s_0 can be uniquely extended to a unital ℓ -group homomorphism $\hat{s}_0 : (K, v) \rightarrow (\mathbb{H}, 1)$. Now if x is any element of K , then $x + c_t - x \in M$ because M is symmetric, and hence

$\hat{s}_0(x + c_t - x) = \hat{s}_0(x) + \hat{s}_0(c_t) - \hat{s}_0(x) = s_0(c_t) = t$ which implies $x + c_t - x = c_t$ so that $x + c_t = c_t + x$.

In other words, we have proved that $(M_t : t \in [0, 1]_{\mathbb{H}})$ has the strong cyclic property, and consequently, M is strong $(\mathbb{H}, 1)$ -perfect. By Theorem 4.2, $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G', (1, 0))$ for some unique (up to isomorphism) ℓ -group G' . Hence $G' \cong G$, where G was defined above, which proves (ii) \Rightarrow (iii).

The implication (iii) \Rightarrow (i) is evident by the note that is just before Theorem 5.7. \square

We note that if M is a local symmetric pseudo MV-algebra with a retractive ideal $\text{Rad}_n(M)$, then M is a lexicographic extension of $\text{Ker}_n(M)$ in the sense described in [HoRa].

Proposition 5.8. *Let $(M_\alpha : \alpha \in A)$ be a system of pseudo MV-algebras and let I_α be a non-trivial normal ideal of M_α , $\alpha \in A$. Set $M = \prod_\alpha M_\alpha$ and $I = \prod_\alpha I_\alpha$. Then I is a retractive ideal of M if and only if every I_α is a retractive ideal of M_α .*

Proof. The set $I = \prod_\alpha I_\alpha$ is a non-trivial normal ideal of M . Then $M/I \cong \prod_\alpha M_\alpha/I_\alpha$ and without loss of generality, we can assume that $M/I = \prod_\alpha M_\alpha/I_\alpha$.

Assume that every I_α is retractive. We denote by π_α the canonical projection of M_α onto M_α/I_α and by $\delta_\alpha : M_\alpha/I_\alpha \rightarrow M_\alpha$ its right inversion i.e. $\pi_\alpha \circ \delta_\alpha = \text{id}_{M_\alpha/I_\alpha}$. Let $\pi : M \rightarrow M/I$ be the canonical projection. If we set $\delta : M/I \rightarrow M$ by $\delta((x_\alpha/I_\alpha)_\alpha) := ((\delta_\alpha(x_\alpha/I_\alpha))_\alpha)$, then we have $\pi \circ \delta = \text{id}_{M/I}$, so that I is retractive.

Conversely, let I be a retractive ideal of M . Let $\pi^\alpha : \prod_\alpha M_\alpha$ be the α -th projection of M onto M_α . We define a mapping $\delta_\alpha : M_\alpha/I_\alpha \rightarrow M_\alpha$ by $\delta_\alpha = \pi^\alpha \circ \delta$ ($\alpha \in A$). Then $\prod_\alpha \pi_\alpha \circ \delta_\alpha(x_\alpha/I_\alpha) = \prod_\alpha \pi_\alpha \circ \pi^\alpha \circ \delta(x_\alpha/I_\alpha)$ which yields $\pi_\alpha \circ \delta_\alpha = \text{id}_{M_\alpha/I_\alpha}$. \square

Corollary 5.9. *Let I be a non-trivial normal ideal of a pseudo MV-algebra M and let α be a cardinal. Then the power I^α is a retractive ideal of the power pseudo MV-algebra M^α if and only if I is a retractive ideal of M .*

6. FREE PRODUCT AND LOCAL PSEUDO MV-ALGEBRAS

In the present section we show that every local pseudo MV-algebra that is a strong $(\mathbb{H}, 1)$ -perfect pseudo MV-algebra has also a representation via a free product. It will generalize results from [DiLe2] known only for local MV-algebras.

Let \mathcal{V} be a class of pseudo MV-algebras and let $\{A_t\}_{t \in T} \subseteq \mathcal{V}$. According to [DvHo1], we say that a \mathcal{V} -coproduct (or simply a *coproduct* if \mathcal{V} is known from the context) of this family is a pseudo MV-algebra $A \in \mathcal{V}$, together with a family of homomorphisms

$$\{f_t : A_t \rightarrow A\}_{t \in T}$$

such that

- (i) $\bigcup_{t \in T} f_t(A_t)$ generates A ;
- (ii) if $B \in \mathcal{V}$ and $\{g_t : A_t \rightarrow B\}_{t \in T}$ is a family of homomorphisms, then there exists a (necessarily) unique homomorphism $h : A \rightarrow B$ such that $g_t = f_t h$ for all $t \in T$.

Coproducts exist for every variety \mathcal{V} of algebra, and are unique. They are designated by $\bigsqcup_{t \in T}^\mathcal{V} A_t$ (or $A_1 \sqcup^\mathcal{V} A_2$ if $T = \{1, 2\}$). If each of the homomorphisms f_t is an embedding, then the coproduct is called the *free product*.

By [DvHo1, Thm 2.3], the free product of any set of non-trivial pseudo MV-algebras exists in the variety of pseudo MV-algebras.

Now let M be a symmetric local pseudo MV-algebra from \mathcal{M} with a unique maximal and normal ideal $I = \text{Ker}_n(M) = \text{Ker}(M)$. Let \mathbb{H} be a subgroup of \mathbb{R} such that $M/I \cong \Gamma(\mathbb{H}, 1)$. Take an ℓ -group G such that $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0)) \cong \langle I \rangle$. Let $N = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$. If I is retractive, then by Theorem 5.7, $M \cong N$, and in this section, we describe this situation using free product of M/I and $\langle I \rangle$. We note that this was already established in [DiLe2, Thm 3.1] but only for MV-algebras. For our generalization, we introduce a weaker form of our free product of M/I and $\langle I \rangle$ which we will denote $M/I \sqcup_w \langle I \rangle$ in the variety of symmetric pseudo MV-algebras from \mathcal{M} and which means that (i) remains and (ii) are changed as follows

- (i*) if $\phi_1 : M/I \rightarrow M/I \sqcup_w \langle I \rangle$ and $\phi_2 : \langle I \rangle \rightarrow M/I \sqcup_w \langle I \rangle$ are injective homomorphisms, then $\phi_1(M/I) \cup \phi_2(\langle I \rangle)$ generates $M/I \sqcup_w \langle I \rangle$,
- (ii*) if $\kappa_1 : M/I \rightarrow A$ and $\kappa_2 : \langle I \rangle \rightarrow A$, where A is a symmetric pseudo MV-algebra from \mathcal{M} , are such homomorphisms that $\kappa_1(a) + \kappa_2(b) = \kappa_2(b) + \kappa_1(a)$, then there is a unique homomorphism $\psi : M/I \sqcup_w \langle I \rangle \rightarrow A$ such that $\psi \circ \phi_1 = \kappa_1$ and $\psi \circ \phi_2 = \kappa_2$.

We note that if M is an MV-algebra, then our notion coincides with the original form of the free product of MV-algebras in the class of MV-algebras.

Theorem 6.1. *Let M be a symmetric local pseudo MV-algebra from \mathcal{M} , $I = \text{Rad}_n(I)$ and $N = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$ for some unital ℓ -subgroup $(\mathbb{H}, 1)$ of $(\mathbb{R}, 1)$ and some ℓ -group G . The following statements are equivalent:*

- (i) $M \cong N$.
- (ii) *The free product $M/I \sqcup_w \langle I \rangle$ in the variety of symmetric pseudo MV-algebras from \mathcal{M} is isomorphic to M .*

Proof. (i) \Rightarrow (ii) Let $M = \Gamma(K, v)$. By Theorem 5.7, $I = \text{Ker}_n(M)$ is a retractive ideal. Define $\phi_1 : M/I \rightarrow \Gamma(\mathbb{H} \overrightarrow{\times} \{0\}, (1, 0)) \subset \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0)) = N$ and $\phi_2 : \langle I \rangle \rightarrow \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0)) \subset \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0)) = N$ as follows: Let s_0 be a unique state on M which is guaranteed by Theorem 3.4. We set $M_t = s_0^{-1}(\{t\})$ for any $t \in [0, 1]_{\mathbb{H}}$. Then $\phi_1(x/I) := (t, 0)$ whenever $x \in M_t$. Since $\langle I \rangle = I \cup I^-$, we set $\phi_2(x) = (0, x)$ if $x \in I$ and $\phi_2(x) = (1, x - 1)$ if $x \in I^-$. From (4.2) of the proof of Theorem 4.2 we see that ϕ_1 and ϕ_2 are injective homomorphisms of pseudo MV-algebras into N . Using again (4.2), we see that $\phi_1(M/I) \cup \phi_2(\langle I \rangle)$ generates N .

Now suppose that there is a symmetric pseudo MV-algebra A from \mathcal{M} and two mutually commuting homomorphisms $\kappa_1 : M/I \rightarrow A$ and $\kappa_2 : \langle I \rangle \rightarrow A = \Gamma(W, w)$, i.e. $\kappa_1(a) + \kappa_2(b) = \kappa_2(b) + \kappa_1(a)$ for all $a \in M/I$ and $b \in \langle I \rangle$. Then $\kappa_1(1/I) = w = \kappa_2(1)$ and w commutes with every $\kappa_1(a)$ and $\kappa_2(b)$.

Claim 1. *Let $a = \kappa_1 \phi_1^{-1}(h, 0)$ with $0 < h < 1$, $h \in H$, and $\epsilon = \kappa_2 \phi_2^{-1}(0, g)$ with $g \in G^+$. Then $\epsilon < a < \epsilon^-$.*

Indeed, by the assumption, from the form of the element a we conclude that it is finite and ϵ and a commute. Then there is an integer $n \geq 1$ such that $n.a = 1$. Since $\epsilon \in \text{Rad}(A)$, we have $n.\epsilon = n\epsilon < 1 = n.a \leq na$ which yields $0 \leq n(a - \epsilon)$, so that $\epsilon < a$. In a similar way we show $\epsilon < a^-$, i.e. $\epsilon < a < \epsilon^-$.

Claim 2. *Let $\alpha = \kappa_1 \circ \phi_1^{-1} : \Gamma(\mathbb{H} \overrightarrow{\times} \{0\}, (1, 0)) \rightarrow A$ and $\beta = \kappa_2 \circ \phi_2^{-1} : \phi_2^{-1}(\langle I \rangle) \rightarrow A$. Passing to the corresponding representing unital ℓ -groups, we will denote by*

$\hat{\alpha}$ and $\hat{\beta}$ the corresponding extensions of α and β to ℓ -homomorphisms of unital ℓ -groups into the unital ℓ -group (G_A, w) such that $\Gamma(G_A, w) = A$. Then $\hat{\alpha}(0, h) + \hat{\beta}(0, g) \geq 0$ for each $h \in \mathbb{H}^+$ and each $g \in G$.

If $h = 0$, the statement is evident. Let $h > 0$. Then $a := \hat{\alpha}(0, h) + \hat{\beta}(0, g) = \hat{\alpha}(h, 0) + \hat{\beta}(0, g^+) + \hat{\beta}(0, g^-)$, where $g^+ = g \vee 0$ and $g^- = g \wedge 0$. Then $a = \hat{\alpha}(h, 0) + \beta(0, g^+) + \beta(1, g^-) - \beta(1, 0)$. From Claim 1, we get $\hat{\alpha}(h, 0) + \beta(0, g^+) \geq \beta(0, -g^-) = \beta(1, 0) - \beta(1, g^-)$ and the claim is proved.

Now we define a mapping $\psi : \mathbb{H} \overrightarrow{\times} G \rightarrow G_A$ by

$$\psi(h, g) = \hat{\alpha}(h, 0) + \hat{\beta}(0, g), \quad (h, g) \in \mathbb{H} \overrightarrow{\times} G.$$

Claim 3. ψ is an ℓ -group homomorphism of unital ℓ -groups.

(a) We have $\psi(0, 0) = 0$ and $\psi(1, 0) = w$. Moreover,

$$\begin{aligned} \psi(h_1, g_1) + \psi(h_2, g_2) &= \hat{\alpha}(h_1, 0) + \hat{\beta}(0, g_1) + \hat{\alpha}(h_2, 0) + \hat{\beta}(0, g_2) \\ &= \hat{\alpha}(h_1, 0) + \hat{\alpha}(h_2, 0) + \hat{\beta}(0, g_1) + \hat{\beta}(0, g_2) \\ &= \hat{\alpha}(h_1 + h_2, 0) + \hat{\beta}(0, g_1 + g_2) \\ &= \psi(h_1 + h_2, g_1 + g_2). \end{aligned}$$

(b) According to Claim 2, we see that $\psi(h, g) \geq 0$ whenever $(h, g) \geq (0, 0)$.

(c) ψ preserves \wedge . For $x := (h_1, g_1) \wedge (h_2, g_2)$, we have three cases (i) $x = (h_1, g_1)$ if $h_1 < h_2$, (ii) $x = (h_1, g_1 \wedge g_2)$ if $h_1 = h_2$, and (iii) $x = (h_2, g_2)$ if $h_2 < h_1$.

In case (i), we have $\psi(h_2, g_2) - \psi(h_1, g_1) = \psi(h_2 - h_1, g_2 - g_1) \geq 0$ by Claim 2. Thus ψ preserves \wedge . In case (ii), we have

$$\begin{aligned} \psi((h_1, g_1) \wedge (h_2, g_2)) &= \psi(h_1, g_1 \wedge g_2) = \hat{\alpha}(h_1, 0) + \hat{\beta}(0, g_1 \wedge g_2) \\ &= \hat{\alpha}(h_1, 0) + \hat{\beta}(0, g_1) \wedge \hat{\beta}(0, g_2) \\ &= (\hat{\alpha}(h_1, 0) + \hat{\beta}(0, g_1)) \wedge (\hat{\alpha}(h_1, 0) + \hat{\beta}(0, g_2)) \\ &= \psi(h_1, g_1) \wedge \psi(h_2, g_2). \end{aligned}$$

Case (iii) follows from (i).

If we restrict ψ to N , then we have

$$\psi(h, g) = (\alpha(h, 0) \oplus \beta(1, g^+)) \odot \beta(1, g^-), \quad (h, g) \in N.$$

Using that ψ is an ℓ -group homomorphism, we have that if $g = g_1 + g_2$, where $g_1 \geq 0$ and $g_2 \leq 0$, then

$$\psi(h, g) = (\alpha(h, 0) \oplus \beta(1, g_1)) \odot \beta(1, g_2).$$

Uniqueness of ψ . If ψ' is another homomorphism from N into A such that $\phi_i \circ \psi = \kappa_i$ for $i = 1, 2$, then $\psi'(0, 0) = \psi(0, 0)$, $\psi'(0, g) = \psi'(\phi_2 \phi_2^{-1}(0, g)) = \phi_2 \kappa_2(0, g) = \psi(0, g)$, $g \in G^+$. $\psi'(h, 0) = \psi'(\phi_1 \phi_1^{-1}(h, 0)) = \phi_1 \kappa_1(h, 0) = \psi(h, 0)$, $h \in [0, 1]_{\mathbb{H}}$.

Using all above steps, we have that the free product $M/I \sqcup_w \langle I \rangle \cong N$. Since $N \cong M$, we have established (ii).

(ii) \Rightarrow (i) From the proof of the previous implication we have that the free product of $\Gamma(\mathbb{H}, 1)$ and $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ is isomorphic to $N = \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$. Since $M/I \cong \Gamma(\mathbb{H}, 1)$ and $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$, we have from (ii) $M \cong N$. \square

7. PSEUDO MV-ALGEBRAS WITH LEXICOGRAPHIC IDEALS

The following notions were introduced in [DFL] only for MV-algebras, and in this section, we extend them for symmetric pseudo MV-algebras and generalize some results from [DFL].

We say that a normal ideal I is (i) *commutative* if $x/I \oplus y/I = y/I \oplus x/I$ for all $x, y \in M$, (ii) *strict* if $x/I < y/I$ implies $x < y$.

For example, (i) if s is a state, then $\text{Ker}(s)$ is a commutative ideal, [Dvu1, Prop 4.1(ix)], (ii) every maximal ideal that is normal is commutative, [Dvu1]. If M is a local symmetric pseudo MV-algebra, Rad_n is a strict ideal.

Now we extend for pseudo MV-algebras the notion of a lexicographic ideal introduced in [DFL] only for MV-algebras. We say that a commutative ideal I of a pseudo MV-algebra M , $\{0\} \neq I \neq M$, is *lexicographic* if

- (i) I is strict,
- (ii) I is retractive,
- (iii) I is prime.

We note that a lexicographic ideal for MV-algebras was defined in [DFL] by (i)–(iii) and

- (iv) $y \leq x \leq y^-$ for all $y \in I$ and all $x \in M \setminus \langle I \rangle$, where $\langle I \rangle$ is the subalgebra of M generated by I .

But since I is strict, we have $y \in I^-$ implies $z < y$ for any $z \in I$. Hence, if $z \notin I$, we have $z/I > x/I = 0/I$ for all $x \in I$ which yields $z > x$. Therefore, $\langle I \rangle = I \cup I^-$ and (iv) holds, and consequently, (iv) from [DFL] is superfluous, and for the definition of a lexicographic ideal of an MV-algebra we need only (i)–(iii).

Let $\text{LexId}(M)$ be the set of lexicographic ideals of M . If we take the MV-algebra M from Example 3.3, we see that $I_1 = \{(0, m, n) : m > 0, n \in \mathbb{Z} \text{ or } m = 0, n \geq 0\}$ and $I_2 = \{(0, 0, n) : n \geq 0\}$ are two unique lexicographic ideals of M and $I_1 \subset I_2$.

Proposition 7.1. *If $I, J \in \text{LexId}(M)$, then $I \subseteq J$ or $J \subseteq I$. In addition, every lexicographic ideal is contained in the radical $\text{Rad}(M)$ of M . If one of the lexicographic ideals is a maximal ideal, then M has a unique maximal ideal of M .*

Proof. Suppose the converse, that is, there are $x \in I \setminus J$ and $y \in J \setminus I$. Then $x/I < y/I$ and $y/J < x/J$ which yields $x < y$ and $y < x$ which is absurd.

Assume that I is a lexicographic ideal of M . If $I = \text{Rad}(M)$, the statement is evident. If there is an element $y \in \text{Rad}(M)$ such that $y \notin I$, then by (ii) $x < y$ for any element $x \in I$, so that $I \subseteq \text{Rad}(M)$.

Let I be any lexicographic ideal of M . We have two cases. (a) I is a maximal ideal of M . We claim M has a unique maximal ideal. Indeed, for any maximal ideal J of M , $J \neq I$, there are $x \in I \setminus J$ and $y \in J \setminus I$ which implies $x < y$ so that $x \in J$ which is a contradiction. Hence, I is a unique maximal ideal of M , then $\text{Rad}(M) = I$ and every lexicographic ideal of M is in $\text{Rad}(M)$. (b) I is not a maximal ideal of M . Let J be an arbitrary maximal ideal of M . There exists $y \in J \setminus I$ which yields $y > x$ for any $x \in I$, so that $x \in J$ and $I \subseteq J$. Hence, again $I \subseteq \text{Rad}(M)$. \square

Remark 7.2. It is clear that if $\text{LexId}(M) \neq \emptyset$ is finite, then $\text{LexId}(M)$ has the greatest element. If $\text{LexId}(M)$ is infinite, we do not know whether $\text{LexId}(M)$ has the greatest element. And if this element exists, is it a maximal ideal of M ?

We note that in Theorem 7.9(1), we show that if M is symmetric from \mathcal{M} and $\text{LexId}(M) \neq \emptyset$, then M is local.

As an interesting corollary we have the following statement.

Corollary 7.3. *If $\text{LexId}(M)$ is non-empty and s is a state on M , then s vanishes on each lexicographic ideal of M .*

Proof. Let I be a lexicographic ideal of M . First let s be an extremal state. Then $\text{Ker}(s)$ is by [Dvu1, Prop 4.3] a maximal ideal. Hence, by Proposition 7.1, we have $I \subseteq \text{Ker}(M) \subseteq \text{Ker}(s)$, so that each extremal state vanishes on I . Therefore, each convex combination of extremal states, and by Krein–Mil’man Theorem, each state on M vanishes on I . \square

A strengthening of the latter corollary for lexicographic pseudo MV-algebras M from \mathcal{M} will be done in Corollary 7.7 showing that then M has a unique state.

Now we present a prototypical examples of a pseudo MV-algebra with lexicographic ideal.

Proposition 7.4. *Let (H, u) be an Abelian linear unital ℓ -group and let G be an ℓ -group. If we set $I = \{(0, g) : g \in G^+\}$, then I is a lexicographic ideal of $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$.*

In addition, M is subdirectly irreducible if and only if G is a subdirectly irreducible ℓ -group.

Proof. It is clear that I is a normal ideal of M as well as it is prime.

We have $x/I = 0/I$ iff $x \in I$. Assume $(0, g)/I < (h, g')/I$. Then $(h, g) \notin I$ that yields $h > 0$ and $(0, g) < (h, g')$. Hence, if $x/I < y/I$, then $(y - x)/I > 0/I$ and $y - x > 0$ and $x < y$.

Since $M/I \cong \Gamma(H \overrightarrow{\times} \{0\}, (u, 0)) \subseteq \Gamma(H \overrightarrow{\times} G, (u, 0))$, we see that I is retractive. Finally, let $y \in I$ and $x \in M \setminus \langle I \rangle$. Then $\langle I \rangle = I \cup I^-$ and $x = (h, g')$ for some h with $0 < h < u$ and $g' \in G$. Then $y = (0, g)$ and hence, $y < x < y^-$.

The statement on subdirect irreducibility follows from the categorical representation of pseudo MV-algebras, Theorem 2.1. \square

Theorem 7.5. *Let M be a symmetric pseudo MV-algebra from \mathcal{M} and let I be a lexicographic ideal of M . Then there is an Abelian linear unital ℓ -group (H, u) and an ℓ -group G such that $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$.*

Proof. Similarly as in the proof of Theorem 5.7, we can assume that $M = \Gamma(K, v)$ for some unital ℓ -group (K, v) . Since I is lexicographic, then I is normal and prime, so that M/I is a linear, and since I is also commutative, M/I is an MV-algebra. There is an Abelian linear unital ℓ -group (H, u) such that $M/I \cong \Gamma(H, u)$.

Let $\pi_I : M \rightarrow M/I$ be the canonical projection. For any $t \in [0, u]_H$, we set $M_t := \pi_I^{-1}(\{t\})$. We assert that $(M_t : t \in [0, u]_H)$ is an (H, u) -decomposition of M . Indeed, (a) let $x \in M_s$ and $y \in M_t$ for $s < t$, $s, t \in [0, u]_H$. Then $\pi_I(x) = s < t < \pi(y)$ and $x < y$ because I is strict. (b) Since π_I is a homomorphism, $M_t^- = M_{u-t} = M_t^\sim$ for each $t \in [0, u]_H$. (c) Let $x \in M_s$ and $y \in M_t$, then $\pi_I(x \oplus y) = \pi_I(x) \oplus \pi_I(y) = s \oplus t$.

In addition, $\langle I \rangle = I \cup I^- = I \cup I^\sim$, $I^- = I^\sim$, and $\langle I \rangle$ is a perfect pseudo MV-algebra. By [DDT, Prop 5.2], there is a unique (up to isomorphism) ℓ -group G such that $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$.

Now we show that $(M_t : t \in [0, u]_H)$ has the strong cyclic property. Being I also retractive, there is a subalgebra M' of M such that $M' \cong M/I$ and $\pi_I(M') = \pi_I(M)$. Then M' is in fact an MV-algebra. For any $t \in [0, u]_H$, there is a unique $c_t \in M_t$ such that $\pi_I(c_t) = t$. We assert that the system of elements $(c_t : t \in [0, u]_H)$ has the following properties: (i) $c_t \in M_t$, (ii) if $s + t \leq u$, then $c_s + c_t \in M$ and $c_s + c_t = c_{s+t}$, (iii) $c_1 = 1$, and (iv) $c_t \in C(K)$ for each $t \in [0, u]_H$; indeed let $x \in K$. Being M symmetric, the element $x + c_t - x \in H$ belongs also to M . Due to the categorical equivalence, Theorem 2.1, the homomorphism π_I can be uniquely extended to a homomorphism $\hat{\pi}_I : (K, v) \rightarrow (H, u)$ of unital ℓ -groups. Hence, $\pi_I(x + c_t - x) = \hat{\pi}_I(x + c_t - x) = \hat{\pi}_I(x) + \hat{\pi}_I(c_t) - \hat{\pi}_I(x) = \pi_I(c_t) = t$ which implies $c_t = x + c_t - x$ and $x + c_t = c_t + x$.

Consequently, M is a strong (H, u) -perfect pseudo MV-algebra. By Theorem 4.2, there is an ℓ -group G' such that $M \cong \Gamma(H \overrightarrow{\times} G', (u, 0))$. By uniqueness (up to isomorphism of ℓ -groups) of G' in Theorem 4.2, we have $G' \cong G$ and consequently $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$. \square

According to the latter theorem and Proposition 7.4, we see that our notion of a lexicographic pseudo MV-algebra for symmetric pseudo MV-algebras from \mathcal{M} coincides with the notion of one defined for MV-algebras in [DFL] as those having at least one lexicographic ideal.

In the following result we compare the class of local pseudo MV-algebras with the class of lexicographic pseudo MV-algebras.

Theorem 7.6. (1) *The class of lexicographic pseudo MV-algebras from \mathcal{M} is strictly included in the class of symmetric local pseudo MV-algebras.*

(2) *The class of symmetric local pseudo MV-algebras with retractive radical is strictly included in the class of lexicographic pseudo MV-algebras from \mathcal{M} .*

Proof. (1) Let M be a lexicographic pseudo MV-algebra from \mathcal{M} . By Theorem 7.5, M is symmetric and it is isomorphic to some $M' := \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is an Abelian unital ℓ -group and G is an ℓ -group. Then the ideal $I = \{(0, g) : g \in G^+\}$ is by Proposition 7.4 a retractive ideal of M' . By Proposition 7.1, we have $I \subseteq \text{Rad}(M') = \text{Rad}_n(M')$. Since I is prime, so is $\text{Rad}_n(M')$ which yields $M'/\text{Rad}_n(M')$ is linearly ordered and semisimple. Hence, $M'/\text{Rad}_n(M')$ is a simple MV-algebra. Therefore, by [Dvu1, Prop 3.3-3.5], $\text{Rad}_n(M)$ is a maximal ideal which yields that M' is local and, consequently M is local.

To show that the class of lexicographic pseudo MV-algebras from \mathcal{M} is strictly included in the class of symmetric local pseudo MV-algebras, we can use an example from the proof of [DFL, Thm 4.7] or the pseudo MV-algebra $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 1))$ that has no lexicographic ideal.

(2) By Theorem 5.7, we get that the class of symmetric local pseudo MV-algebras with retractive radical is strictly included in the class of lexicographic pseudo MV-algebras from \mathcal{M} . Using an example from [DFL, Thm 4.7], we conclude that this inclusion is proper. \square

The latter result entails the following corollary.

Corollary 7.7. *Every lexicographic pseudo MV-algebra from \mathcal{M} admits a unique state.*

Proof. If M is a lexicographic pseudo MV-algebra from \mathcal{M} , by (i) of Theorem 7.6, we see that M is local, that is, it has a unique maximal ideal and this ideal is

normal. Due to a one-to-one relation between extremal states and maximal and normal ideals of M , [Dvu1], we conclude M admits a unique state. \square

The following result gives a new look to Theorem 5.7.

Theorem 7.8. *Let M be a lexicographic symmetric pseudo MV-algebra from \mathcal{M} . The following statements are equivalent:*

- (i) $\text{Rad}_n(M)$ is a lexicographic ideal.
- (ii) M is strongly $(\mathbb{H}, 1)$ -perfect for some unital ℓ -subgroup $(\mathbb{H}, 1)$ of $(\mathbb{R}, 1)$.

Proof. Let $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$ for some Abelian unital ℓ -group (H, u) and an ℓ -group G and let I be a retractive ideal of M such that $M/I \cong \Gamma(H, u)$. By Proposition 7.4, $I \subseteq \text{Rad}_n(M)$.

(i) \Rightarrow (ii) If $\text{Rad}_n(M)$ is a retractive ideal, then $M/\text{Rad}_n(M)$ is a semisimple MV-algebra that is linearly ordered because $\text{Rad}_n(M)$ is a prime normal ideal. Again applying by [Dvu1, Prop 3.4-3.5], $M/\text{Rad}_n(M) \cong \Gamma(H, u)$ and $\Gamma(H, u)$ is isomorphic to some $(\mathbb{H}, 1)$.

(ii) \Rightarrow (i) Since $M/I \cong \Gamma(\mathbb{H}, 1)$, as a consequence of [Dvu1, Prop 3.4-3.5], we get I is a maximal ideal of M . Hence, $I = \text{Rad}_n(M)$ and I is a lexicographic ideal of M and $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, 0))$. \square

We say that a pseudo MV-algebra M from \mathcal{M} is I -representable if I is a lexicographic ideal of M and $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is an Abelian unital ℓ -group such that $M/I \cong \Gamma(H, u)$ and G is an ℓ -group such that $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$; the existences of (H, u) and G are guaranteed by Theorem 7.5.

Theorem 7.9. *The class of lexicographic pseudo MV-algebras from \mathcal{M} is closed under homomorphic images and subalgebras, but it is not closed under direct products.*

Moreover, (1) if N is a homomorphic image of M , then $N \cong \Gamma(H_1 \overrightarrow{\times} G_1, (u_1, 0))$, where (H_1, u_1) and G_1 are homomorphic images of (H, u) and G , respectively.

(2) If N is a subalgebra of M , then $N \cong \Gamma(H_0 \overrightarrow{\times} G_0, (u_0, 0))$, where (H_0, u) and G_0 are subalgebras of (H, u) and G , respectively.

Proof. Let I be a lexicographic ideal of M such that M is I -representable.

(1) Let $f : M \rightarrow N$ be a surjective homomorphism. Then N is symmetric and from \mathcal{M} whilst \mathcal{M} is a variety. If we set $f(I) = \{f(x) : x \in I\}$, then $f(I)$ is a normal ideal of $N = f(M)$ that is also commutative, prime and strict. We claim that $f(I)$ is a retractive ideal, too. Let $\pi_I : M \rightarrow M/I$ be the canonical projection and let $\delta_I : M/I \rightarrow M$ be a homomorphism such that $\pi_I \circ \delta_I = \text{id}_{M/I}$. Let $M_0 = \delta_I(M/I)$ be a subalgebra of M that is isomorphic to M/I . If we define $\hat{f} : M/I \rightarrow N/f(I)$ by $\hat{f}(x/I) = f(x)/f(I)$, then \hat{f} is a well-defined homomorphism such that $\hat{f} \circ \pi_I = \pi_{f(I)} \circ f$. Set $N_0 = f(M_0)$ and let f_{M_0} be the restriction of f onto M_0 . We define $\delta_{f(I)} : N/f(I) \rightarrow N$ via $\delta_{f(I)}(f(x)/f(I)) := f_{M_0}(\delta_I(x/I))$; then $\delta_{f(I)}$ is a well-defined homomorphism such that $\delta_{f(I)}(N/f(I)) = N_0$ and $f_{M_0} \circ \delta_I = \delta_{f(I)} \circ \hat{f}$. Hence,

$$\begin{aligned} \pi_{f(I)} \circ \delta_{f(I)}(f(x)/f(I)) &= \pi_{f(I)} \circ f_{M_0} \circ \delta_I(x/I) \\ &= \hat{f} \circ \pi_I \circ \delta_I(x/I) = \hat{f}(x/I) \\ &= f(x)/f(I) \end{aligned}$$

that proves $f(I)$ is a retractive ideal of N .

Take the unital representation of pseudo MV-algebras given by Theorem 2.1, and let $N \cong \Gamma(K, v)$ and let $f : (H \overrightarrow{\times} G, (u, 0)) \rightarrow (K, v)$ be a surjective homomorphism of unital ℓ -groups. Let $f_1(h) = f(h, 0)$, $h \in H$, and $f_2(g) = f(0, g)$, $g \in G$. If we set $H_1 := f_1(H)$, $u_1 = f(u, 0)$, and $G_1 := f_2(G)$. Then $N \cong \Gamma(H_1 \overrightarrow{\times} G_1, (u_1, 0))$.

(2) Let N be a subalgebra of M . Then N is symmetric and belongs to \mathcal{M} . We set $J := N \cap I$. Then J is a normal ideal of N that is also commutative and prime. It is strict, too, because if $x \in N$ and $x \notin J$, then $x \notin I$ and $x > y$ for any $y \in J$ and consequently, for any $y \in J$. Then N/J can be embedded into M/I by a mapping $i_J(x/J) := x/I$ ($x \in N$) and if $i_0(x) = x$, $x \in N$, then $\pi_I \circ i_0 = i_J \circ \pi_J$. Let $M_0 := \delta_I(M/I)$ and $N_0 := M_0 \cap N$. Then $\delta_I(N/I) \in N_0$; indeed, if there is $x \in N_0$ such that $\delta_I(x/I) \notin N_0$, then $\pi_I \circ \delta_I(x/I) = x/I \notin N_0/I$. Define $\delta_J : N/J \rightarrow N$ by $\delta_J(x/J) = i_J^{-1} \circ \delta_I(x/I)$. Since $i_I^{-1} \circ \pi_I(x) = \pi_J \circ i_0^{-1}(x)$, $x \in N$, then

$$\begin{aligned} \pi_J \circ \delta_J(x/J) &= \pi_J \circ i_0^{-1} \circ \delta_I(x/I) \\ &= i_I^{-1} \circ \pi_I \circ \delta_I(x/I) = i_I^{-1}(x/I) = x/J. \end{aligned}$$

The rest follows the analogous steps as the end of (1).

(3) According to Corollary 7.7, every lexicographic pseudo MV-algebra M admits a unique state. But the pseudo MV-algebra $M \times M$ admits two extremal states, and therefore, $M \times M$ is not lexicographic. \square

We note that in case (3) of latter Theorem if I is a lexicographic ideal of M , then $I \times I$ is by Proposition 5.8 a retractive ideal but not lexicographic.

8. CATEGORICAL REPRESENTATION OF STRONG (H, u) -PERFECT PSEUDO MV-ALGEBRAS

In this section, we establish the categorical equivalence of the category of strong (H, u) -perfect pseudo MV-algebras with the variety of ℓ -groups. This extends the categorical representation of strong n -perfect pseudo MV-algebras from [Dvu3] and of \mathbb{H} -perfect pseudo MV-algebras from [Dvu4] with the variety of ℓ -groups. In what follows, we follow the ideas of [Dvu4, Sec 5] and to be self-contained we repeat them mutatis mutandis.

Let $\mathcal{SPP}_s\mathcal{MV}_{H,u}$ be the category of strong (H, u) -perfect pseudo MV-algebras whose objects are strong (H, u) -perfect pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Now let \mathcal{G} be the category whose objects are ℓ -groups and morphisms are homomorphisms of ℓ -groups.

Define a mapping $\mathcal{M}_{H,u} : \mathcal{G} \rightarrow \mathcal{SPP}_s\mathcal{MV}_{H,u}$ as follows: for $G \in \mathcal{G}$, let

$$\mathcal{M}_{H,u}(G) := \Gamma(H \overrightarrow{\times} G, (u, 0))$$

and if $h : G \rightarrow G_1$ is an ℓ -group homomorphism, then

$$\mathcal{M}_{H,u}(h)(t, g) = (t, h(g)), \quad (t, g) \in \Gamma(H \overrightarrow{\times} G, (u, 0)).$$

It is easy to see that $\mathcal{M}_{H,u}$ is a functor.

Proposition 8.1. *$\mathcal{M}_{H,u}$ is a faithful and full functor from the category \mathcal{G} of ℓ -groups into the category $\mathcal{SPP}_s\mathcal{MV}_{H,u}$ of strong (H, u) -perfect pseudo MV-algebras.*

Proof. Let h_1 and h_2 be two morphisms from G into G' such that $\mathcal{M}_{H,u}(h_1) = \mathcal{M}_{H,u}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for each $g \in G^+$, consequently $h_1 = h_2$.

To prove that $\mathcal{M}_{H,u}$ is a full functor, suppose that f is a morphism from a strong (H, u) -perfect pseudo MV-algebra $\Gamma(H \overrightarrow{\times} G, (u, 0))$ into some $\Gamma(H \overrightarrow{\times} G_1, (u, 0))$. Then $f(0, g) = (0, g')$ for a unique $g' \in G'^+$. Define a mapping $h : G^+ \rightarrow G'^+$ by $h(g) = g'$ iff $f(0, g) = (0, g')$. Then $h(g_1 + g_2) = h(g_1) + h(g_2)$ if $g_1, g_2 \in G^+$. Assume now that $g \in G$ is arbitrary. Then $g = g^+ - g_1$, where $g^+ = g \vee 0$ and $g^- = -(g \wedge 0)$, and $g = -g^- + g^+$. If $g = g_1 - g_2$, where $g_1, g_2 \in G^+$, then $g^+ + g_2 = g^- + g_1$ and $h(g^+) + h(g_2) = h(g^-) + h(g_1)$ which shows that $h(g) = h(g_1) - h(g_2)$ is a well-defined extension of h from G^+ onto G .

Let $0 \leq g_1 \leq g_2$. Then $(0, g_1) \leq (0, g_2)$, which means h is a mapping preserving the partial order.

We have yet to show that h preserves \wedge in G , i.e., $h(a \wedge b) = h(a) \wedge h(b)$ whenever $a, b \in G$. Let $a = a^+ - a^-$ and $b = b^+ - b^-$, and $a = -a^- + a^+$, $b = -b^- + b^+$. Since $h((a^+ + b^-) \wedge (a^- + b^+)) = h(a^+ + b^-) \wedge h(a^- + b^+)$. Subtracting $h(b^-)$ from the right hand and $h(a^-)$ from the left hand, we obtain the statement in question.

Finally, we have proved that h is a homomorphism of ℓ -groups, and $\mathcal{M}_{H,u}(h) = f$ as claimed. \square

We note that by a *universal group* for a pseudo MV-algebra M we mean a pair (G, γ) consisting of an ℓ -group G and a G -valued measure $\gamma : M \rightarrow G^+$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ is defined in M) such that the following conditions hold: (i) $\gamma(M)$ generates G . (ii) If K is a group and $\phi : M \rightarrow K$ is an K -valued measure, then there is a group homomorphism $\phi^* : G \rightarrow K$ such that $\phi = \phi^* \circ \gamma$.

Due to [Dvu2], every pseudo MV-algebra admits a universal group, which is unique up to isomorphism, and ϕ^* is unique. The universal group for $M = \Gamma(G, u)$ is (G, id) where id is the embedding of M into G .

Let \mathcal{A} and \mathcal{B} be two categories and let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. Suppose that g, h be two functors from \mathcal{B} to \mathcal{A} such that $g \circ f = id_{\mathcal{A}}$ and $f \circ h = id_{\mathcal{B}}$, then g is a *left-adjoint* of f and h is a *right-adjoint* of f .

Proposition 8.2. *The functor $\mathcal{M}_{H,u}$ from the category \mathcal{G} into $SPP_s\mathcal{MV}_{H,u}$ has a left-adjoint.*

Proof. We show, for a strong (H, u) -perfect pseudo MV-algebra M with an (H, u) -decomposition $(M_t : t \in [0, u]_H)$ and an (H, u) -strong cyclic family $(c_t : t \in [0, u]_H)$ of elements of M , there is a universal arrow (G, f) , i.e., G is an object in \mathcal{G} and f is a homomorphism from the pseudo MV-algebra M into $\mathcal{M}_{H,u}(G)$ such that if G' is an object from \mathcal{G} and f' is a homomorphism from M into $\mathcal{M}_{H,u}(G')$, then there exists a unique morphism $f^* : G \rightarrow G'$ such that $\mathcal{M}_{H,u}(f^*) \circ f = f'$.

By Theorem 4.2, there is a unique (up to isomorphism of ℓ -groups) ℓ -group G such that $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$. By [Dvu2, Thm 5.3], $(H \overrightarrow{\times} G, \gamma)$ is a universal group for M , where $\gamma : M \rightarrow \Gamma(H \overrightarrow{\times} G, (u, 0))$ is defined by $\gamma(a) = (t, a - c_t)$, if $a \in M_t$. \square

Define a mapping $\mathcal{P}_{H,u} : SPP_s\mathcal{MV}_{H,u} \rightarrow \mathcal{G}$ via $\mathcal{P}_{H,u}(M) := G$ whenever $(H \overrightarrow{\times} G, f)$ is a universal group for M . It is clear that if f_0 is a morphism from the pseudo MV-algebra M into another one N , then f_0 can be uniquely extended to an ℓ -group homomorphism $\mathcal{P}_{H,u}(f_0)$ from G into G_1 , where $(H \overrightarrow{\times} G_1, f_1)$ is a universal group for the strong (H, u) -perfect pseudo MV-algebra N .

Proposition 8.3. *The mapping $\mathcal{P}_{H,u}$ is a functor from the category $SPP_s\mathcal{MV}_{H,u}$ into the category \mathcal{G} which is a left-adjoint of the functor $\mathcal{M}_{H,u}$.*

Proof. It follows from the properties of the universal group. \square

Now we present the basic result of this section on a categorical equivalence of the category of strong (H, u) -perfect pseudo MV-algebras and the category of \mathcal{G} .

Theorem 8.4. *The functor $\mathcal{M}_{H,u}$ defines a categorical equivalence of the category \mathcal{G} and the category $SPP_s\mathcal{MV}_{H,u}$ of strong (H, u) -perfect pseudo MV-algebras.*

In addition, suppose that $h : \mathcal{M}_{H,u}(G) \rightarrow \mathcal{M}_{H,u}(G')$ is a homomorphism of pseudo MV-algebras, then there is a unique homomorphism $f : G \rightarrow G'$ of ℓ -groups such that $h = \mathcal{M}_{H,u}(f)$, and

- (i) *if h is surjective, so is f ;*
- (ii) *if h is injective, so is f .*

Proof. According to [MaL, Thm IV.4.1], it is necessary to show that, for a strong (H, u) -perfect pseudo MV-algebra M , there is an object G in \mathcal{G} such that $\mathcal{M}_{H,u}(G)$ is isomorphic to M . To show that, we take a universal group $(H \overrightarrow{\times} G, f)$. Then $\mathcal{M}_{H,u}(G)$ and M are isomorphic. \square

An important kind of ℓ -groups are doubly transitive ℓ -groups; for more details on them see e.g. [Gla]. Every such an ℓ -group generates the variety of ℓ -groups, [Gla, Lem 10.3.1]. The notion of doubly transitive unital ℓ -group (G, u) was introduced and studied in [DvHo], and according to [DvHo, Cor 4.9], the pseudo MV-algebra $\Gamma(G, u)$ generates the variety of pseudo MV-algebras.

An example of a doubly transitive permutation ℓ -group is the system of all automorphisms, $\text{Aut}(\mathbb{R})$, of the real line \mathbb{R} , or the next example:

Let $u \in \text{Aut}(\mathbb{R})$ be the translation $tu = t + 1$, $t \in \mathbb{R}$, and

$$\text{BAut}(\mathbb{R}) = \{g \in \text{Aut}(\mathbb{R}) : \exists n \in \mathbb{N}, u^{-n} \leq g \leq u^n\}.$$

Then $(\text{BAut}(\mathbb{R}), u)$ is a doubly transitive unital ℓ -permutation group, and it is a generator of the variety of pseudo MV-algebras $\mathcal{P}_s\mathcal{MV}$. In addition, $\Gamma(\text{BAut}(\mathbb{R}), u)$ is a stateless pseudo MV-algebra.

The proof of the following statement is practically the same as that of [Dvu4, Thm 5.6] and therefore, we omit it here.

Theorem 8.5. *Let G be a doubly transitive ℓ -group. Then the variety generated by $SPP_s\mathcal{MV}_{H,u}$ coincides with the variety generated by $\mathcal{M}_{H,u}(G)$.*

9. WEAK (H, u) -PERFECT PSEUDO MV-ALGEBRAS

In this section, we will study another kind of (H, u) -perfect pseudo MV-algebras, called weak (H, u) -perfect pseudo MV-algebras. Their prototypical examples are pseudo MV-algebras of the form $\Gamma(H \overrightarrow{\times} G, (1, b))$, where (H, u) is an Abelian unital ℓ -group, G is an ℓ -group and $b \in G$. Such pseudo MV-algebras were studied for the case $(H, u) = (\mathbb{H}, 1)$ in [Dvu4].

Let (H, u) be an Abelian unital ℓ -group. We say that a pseudo MV-algebra $M \cong \Gamma(K, v)$ with an (H, u) -decomposition $(M_t : t \in [0, u]_H)$ is *weak* if there is a system $(c_t : t \in [0, u]_H)$ of elements of M such that (i) $c_0 = 0$, (ii) $c_t \in C(K) \cap M_t$, for any $t \in [0, u]_H$, and (iii) $c_{v+t} = c_v + c_t$ whenever $v + t \leq u$.

We notice that in contrast to the strong cyclic property, we do not assume $c_1 = 1$. In addition, a weak (H, u) -perfect pseudo MV-algebra M is strong iff $c_1 = 1$.

Example 9.1. Let (H, u) be an Abelian unital ℓ -group. The pseudo MV-algebra $M = \Gamma(H \overrightarrow{\times} G, (u, b))$, where $b \in G$, $M_t = \{(t, g) : (t, g) \in M\}$, $t \in [0, u]_H$ form an (H, u) -decomposition of M , is a weak pseudo MV-algebra setting $c_t = (t, 0)$, $t \in [0, u]_H$.

Proof. We have to verify that $(M_t : t \in [0, u]_H)$ is an (H, u) -decomposition. To show that it is enough to verify (b) of Definition 3.1, i.e. $M_t^- = M_{u-t} = M_t^\sim$ for each $t \in [0, u]_H$. Let $(t, g) \in M_t$. Then $(t, g)^- = (u, b) - (t, g) = (u - t, b - g)$. If we choose (t, g_0) , where $g_0 = b + g - b$, then $(t, g_0)^\sim = -(t, g_0) + (u, b) = (-t + u, -g_0 + b) = (u - t, b - g) = (t, g)^-$ which yields $(t, g)^- \in M_t^\sim$, that is $M_t^- \subseteq M_t^\sim$. Dually we show $M_t^\sim \subseteq M_t^-$. Then $M_t^- = M_{u-t} = M_t^\sim$. \square

Whereas every strong (H, u) -perfect pseudo MV-algebra is symmetric, weak ones are not necessarily symmetric.

For example, the pseudo MV-algebra $\Gamma(H \overrightarrow{\times} G, (u, b))$, where $b > 0$ and $b \notin C(G)$ and $M_t := \{(t, g) \in \Gamma(H \overrightarrow{\times} G, (u, b))\}$ for each $t \in [0, u]_H$, is weak (H, u) -perfect but neither strong (H, u) -perfect nor symmetric.

We note that M_0 is a unique maximal and normal ideal of M . This ideal is retractive iff M is strongly (H, u) -perfect. For example, let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 1))$. Then M is weakly $(\mathbb{Z}, 2)$ -perfect that is not strongly $(\mathbb{Z}, 2)$ -perfect, and $M_0 = \{(0, n) : n \geq 0\}$, $M_1 = \{(1, n) : n \in \mathbb{Z}\}$, $M_2 = \{(2, n) : n \leq 1\}$, $M/M_0 \cong \Gamma(\frac{1}{2}\mathbb{Z}, 1)$ and it has no isomorphic copy in M . In addition, M_0 is not retractive.

We notice that even a pseudo MV-algebra of the form $\Gamma(H \overrightarrow{\times} G, (u, b))$ with $b \neq 0$ can be strongly (H, u) -perfect. Indeed, let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 2))$. This MV-algebra is isomorphic with the MV-algebra $M_1 := \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 0))$. In fact, the mapping $\theta : M_1 \rightarrow M$ defined by $\theta(0, n) = (0, n)$, $\theta(1, n) = (1, n + 1)$ and $\theta(2, n) = (2, n + 2)$ is an isomorphism in question. In addition, $M_0 = \{(0, n) : n \geq 0\}$ is a retractive ideal and a lexicographic ideal of M ; $M/M_0 = \Gamma(\frac{1}{2}\mathbb{Z}, 1)$ and its isomorphic copy in M is the subalgebra $\{(0, 0), (1, 1), (2, 2)\}$.

The next result is a representation of weak (H, u) -perfect pseudo MV-algebras by lexicographic product.

Theorem 9.2. Let M be a weak \mathbb{H} -perfect pseudo MV-algebra which is not strong. Then there is a unique (up to isomorphism) ℓ -group G with an element $b \in G$, $b \neq 0$, such that $M \cong \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, b))$.

Proof. Assume $M = \Gamma(K, v)$ for some unital ℓ -group (H, u) is a weak pseudo MV-algebra with a (H, u) -decomposition $(M_t : t \in [0, u]_H)$. Since by (vi) of Theorem 3.2 we have $M_0 + M_0 = M_0$, in the same way as in the proof of Theorem 4.2, there exists an ℓ -group G such that $G^+ = M_0$ and G is a subgroup of K .

Since M is not strong, then $c_1 < 1 =: u$. Set $b = 1 - c_1 \in M_0 \setminus \{0\}$, and define a mapping $\phi : M \rightarrow \Gamma(\mathbb{H} \overrightarrow{\times} G, (1, b))$ as follows

$$\phi(x) = (t, x - c_t)$$

whenever $x \in M_t$; we note that the subtraction $x - c_t$ is defined in the ℓ -group K . Using the same way as that in (4.2), we can show that ϕ is a well-defined mapping.

We have (1) $\phi(0) = (0, 0)$, (2) $\phi(1) = (1, 1 - c_1) = (1, b)$, (3) $\phi(c_t) = (t, 0)$, (4) $\phi(x^\sim) = (1 - t, -x + u - c_{1-t}) = (1 - t, -x + b + c_t)$, $\phi(x)^\sim = -\phi(x) + (1, b) = -(t, x - c_t) + (1, b) = (1 - t, -x + b + c_t)$, and similarly (5) $\phi(x^-) = \phi(x)^-$.

Following ideas of the proof of Theorem 4.2, we can prove that ϕ is an injective and surjective homomorphism of pseudo MV-algebras as was claimed. \square

It is worthy of reminding that Theorem 9.2 is a generalization of Theorem 4.2, because Theorem 4.2 in fact follows from Theorem 9.2 when we have $b = 0$. This happens if $c_1 = 1$.

Also in an analogous way as in [Dvu4], we establish a categorical equivalence of the category of weak (H, u) -perfect pseudo MV-algebras with the category of ℓ -groups G with a fixed element $b \in G$.

Let $\mathcal{WPP}_s\mathcal{MV}_{H,u}$ be the category of weak (H, u) -perfect pseudo MV-algebras whose objects are weak (H, u) -perfect pseudo MV-algebras and morphisms are homomorphisms of pseudo MV-algebras. Similarly, let \mathcal{L}_b be the category whose objects are couples (G, b) , where G is an ℓ -group and b is a fixed element from G , and morphisms are ℓ -homomorphisms of ℓ -groups preserving fixed elements b .

Define a mapping $\mathcal{F}_{H,u}$ from the category \mathcal{L}_b into the category $\mathcal{WPP}_s\mathcal{MV}_{H,u}$ as follows:

Given $(G, b) \in \mathcal{L}_b$, we set

$$\mathcal{F}_{H,u}(G, b) := \Gamma(H \overrightarrow{\times} G, (u, b)),$$

and if $h : (G, b) \rightarrow (G_1, b_1)$, then

$$\mathcal{F}_{H,u}(h)(t, g) = (t, h(g)), \quad (t, g) \in \Gamma(H \overrightarrow{\times} G, (u, b)).$$

It is easy to see that $\mathcal{F}_{H,u}$ is a functor.

In the same way as the categorical equivalence of strong (H, u) -perfect pseudo MV-algebras was proved in the previous section, we can prove the following theorem.

Theorem 9.3. *The functor $\mathcal{F}_{H,u}$ defines a categorical equivalence of the category \mathcal{L}_b and the category $\mathcal{WPP}_s\mathcal{MV}_{H,u}$ of weak (H, u) -perfect pseudo MV-algebras.*

Finally, we present addition open problems.

Problem 9.4. (1) Find an equational basis for the variety generated by the set $\mathcal{SPP}_s\mathcal{MV}_{H,u}$. For example, if $(H, u) = (\mathbb{Z}, 1)$ the basis is $2.x^2 = (2.x)^2$, see [DDT, Rem 5.6], and the case $(H, u) = (\mathbb{Z}, n)$ was described in [Dvu3, Cor 5.8].

(2) Find algebraic conditions that entail that a pseudo MV-algebra is of the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$, where (H, u) is a unital ℓ -group not necessary Abelian.

10. CONCLUSION

In the paper we have established conditions when a pseudo MV-algebra M is an interval in some lexicographic product of an Abelian unital ℓ -group (H, u) and an ℓ -group G not necessarily Abelian, i.e. $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$. To show, that we have introduced strong (H, u) -perfect pseudo MV-algebras as those pseudo MV-algebras that can be split into comparable slices indexed by the elements from the interval $[0, u]_H$. For them we have established a representation theorem, Theorem 4.2, and we have shown that the category of strong (H, u) -perfect pseudo MV-algebras is categorically equivalent to the variety of ℓ -groups, Theorem 8.4.

We have shown that our aim can be solved also introducing so-called lexicographic ideals. We establish their properties and Theorem 7.5 gives also a representation of a pseudo MV-algebra in the form $\Gamma(H \overrightarrow{\times} G, (u, 0))$. We show that every lexicographic pseudo MV-algebra is always local, Theorem 7.6.

Finally, we have studied and represented weak (H, u) -perfect pseudo MV-algebras as those that they have a form $\Gamma(H \overrightarrow{\times} G, (u, g))$ where $g \in G$ is not necessary the zero element, Theorem 9.2.

The present study has opened a door into a large class of pseudo MV-algebras and formulated new open questions, and we hope that it stimulate a new research on this topic.

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